Journal of Statistical Physics, Vol. 71, Nos. 5/6, 1993

The Spin-1/2 *XXZ* Heisenberg Chain, the Quantum Algebra $U_q[sl(2)]$, and Duality Transformations for Minimal Models

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Received October 12, 1992; final December 11, 1992

The finite-size scaling spectra of the spin-1/2 *XXZ* Heisenberg chain with toroidal boundary conditions and an even number of sites provide a projection mechanism yielding the spectra of models with a central charge $c<1$, including the unitary and nonunitary minimal series. Taking into account the half-integer angular momentum sectors—which correspond to chains with an odd number of sites—in many cases leads to new spinor operators appearing in the projected systems. These new sectors in the *XXZ* chain correspond to new types of frustration lines in the projected minimal models. The corresponding new boundary conditions in the Hamiltonian limit are investigated for the Ising model and the 3-state Potts model and are shown to be related to duality transformations which are an additional symmetry at their self-dual critical point. By different ways of projecting systems we find models with the same central charge sharing the same operator content and modular invariant partition function which, however, differ in the distribution of operators into sectors and hence in the physical meaning of the operators involved. Related to the projection mechanism in the continuum there are remarkable symmetry properties of the finite *XXZ* chain. The observed degeneracies in the energy and momentum spectra are shown to be the consequence of intertwining relations involving $U_q[sl(2)]$ quantum algebra transformations.

KEY WORDS: *XXZ* Heisenberg chain; scaling limit; conformal invariance; minimal models; quantum algebras; intertwiners; duality; toroidal boundary conditions.

1. INTRODUCTION

Recently, the spin-1/2 *XXZ* Heisenberg chain has attracted interest since it was found[1] that the finite-size scaling limit spectra of the chain with

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0022-4715/93/0600-0923$07.00/0

toroidal boundary conditions and an even number of sites allow a projection mechanism that yields the spectra of minimal unitary models with central charge $c < 1$. The projection mechanism for the continuum models is based on the Feigin–Fuchs construction[2] of the character functions of the Virasoro algebra with central charge $c < 1$ from the character functions with $c = 1$. By taking differences of partition functions in the finite-size scaling limit of the XXZ Hamiltonian with toroidal boundary conditions (which corresponds to a free boson field theory with central charge $c = 1$), one obtains partition functions for models with a central charge $c < 1$. This can be done in various ways, yielding two classes of models, which we call the R and the L models, and which in turn are each divided into infinite series of models labeled by a positive integer, also denoted by R and L, respectively.

But it has also been realized[1] that the projection mechanism has a meaning for *finite* systems as well. In many cases, huge degeneracies in the spectra allow an analogous operation on the finite-size spectra, where instead of differences of partition functions one considers differences of spectra (where this means differences in terms of sets of eigenvalues). This means that one throws away all degenerate levels, keeping only singlets. In this way, one can obtain, for instance, the *exact* finite-size spectra of the Ising and 3-state Potts quantum chains with N sites from the spectra of the XXZ chain with suitably chosen anisotropy and boundary conditions with $2N$ sites.[1]

Although the spin-1/2 XXZ Heisenberg chain with toroidal boundary conditions is not invariant under the quantum algebra $U_q[sl(2)]$, it has been realized[3] that part of the degeneracies related to the projection mechanism which were observed for finite chains[1] can be explained by investigation of the action of $U_q[sl(2)]$ transformations. In this paper, we show that this is indeed true for *all* the degeneracies observed in ref. 1 and that in this way one can explain the degeneracy of both the energy and the momentum eigenvalues of the respective levels. This answers part of the questions left open in ref. 1.

Similar projection mechanisms to the one outlined above have been found in the XXZ Heisenberg chain with free boundary conditions and appropriately chosen surface fields at the ends of the chain[4–6] (including a special choice with a $U_q[sl(2)]$-invariant Hamiltonian) and for a spin-1 quantum chain[7] which allows one to extract the spectra of systems belonging to the minimal superconformal series. Recently, similar structures have been observed[8] by investigating the spectrum of the 3-state superintegrable chiral Potts model, which is related to a spin-1 XXZ chain with anisotropy parameter $\gamma = \pi/3$.

In this paper we generalize the projection mechanism of ref. 1. The class of systems obtained through the projection procedure is enlarged to obtain models for all values $c<1$ of the central charge, including the non-unitary minimal series. Our main emphasis, however, lies in the investigation of the same type of projection mechanism, but now applied to half-integer angular momentum sectors, i.e., we use the spectra of the *XXZ* chain with an odd number of sites. This results in the appearance of new spinor operators in two classes of models and it therefore corresponds to new types of frustration lines in the minimal models which are obtained by the projection procedure. As explicit examples we study the Hamiltonian limit of the Ising model and the 3-state Potts model. Here we find interesting new boundary conditions which have not been considered so far and which turn out to be related to duality transformations.

Of special interest are also those models where the same operator content can be obtained from an even and an odd number of sites. Our numerical data for finite chains indicate that in these cases it is not necessary to consider even and odd lengths in the finite chains separately. This observation agrees with results obtained for open chains.[6]

Another result of our investigation is that we find systems which have the same central charge, the same operator content, and the same modular invariant partition function, but which differ in the distribution of operators into sectors defined by the global symmetries of the model. This means that the physical significance of these operators is different and therefore the operator content and the modular invariant partition function alone are not sufficient to characterize completely the universality class of a critical system. One must take into account also the possible discrete symmetries that are not determined by the partition function alone.

Our paper is organized as follows. In Section 2 we show how one can obtain the spectra of $c<1$ systems by projection from the finite-size scaling spectra of the *XXZ* Heisenberg chain in the continuum following the scheme set up in ref. 1, our emphasis, however, lying on the half-integer angular momentum sectors, which in this context have not been considered previously. The R and L models are defined and the operator content is given for the $R=1,2$ and $L=1,2$ models. We illustrate the projection mechanism by means of explicit examples for the $R=1$ and $L=1$ models, where one obtains additional sectors from the half-integer angular momentum sectors involving new spinor operators in the projected systems. Section 3 deals with the implications of the projection mechanism for finite chains. All the degeneracies observed here and in ref. 1 can be explained using the representation theory of the quantum algebra $U_q[sl(2)]$ (see Appendix A). In Section 4 we give an interpretation of the new sectors obtained from half-integer spin sectors of the *XXZ* chain. Here we focus on the minimal models in the $R=1$ series. As two explicit examples we consider the Hamiltonian limit of the Ising model and of the 3-state Potts model. We show that for these cases one has to choose a new type of boundary condition which is related to duality transformations, which in a sense act as a "square root" of the translation operator. Furthermore, we present some numerical data for a model with central charge $c = -22/5$, which belongs to that class of models for which the same operator content is obtained from the scaling limit using an even and an odd number of sites in the XXZ Heisenberg chain.

In the two appendices we show that $U_q[sl(2)]$ quantum group transformations explain the observed degeneracies between the spectra of the finite XXZ Heisenberg chain with *different* appropriately chosen toroidal boundary conditions. For this purpose, following the ideas of ref. 3, we establish intertwining relations between different sectors of XXZ Hamiltonians with different toroidal boundary conditions using the quantum algebra generators and make use of the known structure of the irreducible representations of $U_q[sl(2)]$. Going beyond the results of ref. 3, this discussion also includes the equality of the momenta of the levels concerned. Furthermore, we give a short reminder of duality transformations for the Ising and the 3-state Potts quantum chains.

2. PROJECTION MECHANISM IN THE FINITE-SIZE SCALING LIMIT

Let us consider the spin-1/2 XXZ Heisenberg chain with a general toroidal boundary condition α defined by the Hamiltonian[9,10]

$$H(q, \alpha, N) = -\tfrac{1}{2}\left\{ \sum_{j=1}^{N-1} \left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + (q+q^{-1})\, \sigma_j^z \sigma_{j+1}^z\right) + \alpha \sigma_N^+ \sigma_1^- + \alpha^{-1} \sigma_N^- \sigma_1^+ + (q+q^{-1})\, \sigma_N^z \sigma_1^z \right\} \tag{2.1}$$

acting on a Hilbert space $\mathscr{H}(N) \cong (\mathbb{C}^2)^{\otimes N}$. Here N denotes the number of sites, q and α are (for the moment) arbitrary complex numbers, and $\sigma_j^\pm = \sigma_j^x \pm i\sigma_j^y$, where σ_j^x, σ_j^y, and σ_j^z are the Pauli matrices acting on the jth site of the chain. Note that the Hamiltonian $H(q, \alpha, N)$ (which is related to the six-vertex model in the presence of a horizontal electric field[11-13]) is Hermitian if $|\alpha| = 1$ and if either q is real or $|q| = 1$.

For arbitrary values of the parameters q and α, the Hamiltonian (2.1) commutes with the total spin (or "charge") operator

$$S^z = \tfrac{1}{2} \sum_{j=1}^{N} \sigma_j^z \tag{2.2}$$

Therefore, we can split the Hilbert space $\mathscr{H}(N)$ into a direct sum of $2N+1$ spaces with fixed value Q of S^z,

$$\mathscr{H}(N) = \bigoplus_{Q=-N/2}^{Q=N/2} \mathscr{H}_Q(N) \tag{2.3}$$

where Q runs over the integer (half-integer) numbers depending on N being even (odd), respectively. We denote by $\mathscr{P}_Q$ the projectors onto the subspaces $\mathscr{H}_Q$, $-N/2 \leqslant Q \leqslant N/2$. The dimension of $\mathscr{H}_Q(N)$ is given by

$$\binom{N}{Q+N/2}, \qquad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The Hamiltonian (2.1) also commutes with a translation operator $T(\alpha, N)$, which can be represented as an operator on $\mathscr{H}(N)$ in terms of Pauli matrices as follows:

$$T(\alpha, N) = \alpha^{-\sigma_1^z/2} \cdot \prod_{j=1}^{N-1} P_j = \prod_{j=1}^{N-1} P_j \cdot \alpha^{-\sigma_N^z/2} = \prod_{j=1}^{N-1} \tilde{P}_j(\alpha, N) \cdot \alpha^{-S^z/N} \tag{2.4}$$

Here the operators P_j (which permute the observables on sites j and $j+1$ in an obvious way) and $\tilde{P}_j$ are defined by

$$P_j = \tfrac{1}{4}[\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + 2(\sigma_j^z \sigma_{j+1}^z + 1)] = P_j^\dagger = P_j^{-1} \tag{2.5}$$

$$\begin{aligned}\tilde{P}_j(\alpha, N) &= \alpha^{-\sigma_j^z/2N} P_j \alpha^{\sigma_j^z/2N} = \alpha^{\sigma_{j+1}^z/2N} P_j a^{-\sigma_{j+1}^z/2N} \\ &= \tfrac{1}{4}[\alpha^{-1/N} \sigma_j^+ \sigma_{j+1}^- + \alpha^{1/N} \sigma_j^- \sigma_{j+1}^+ + 2(\sigma_j^z \sigma_{j+1}^z + 1)]\end{aligned} \tag{2.6}$$

where $j = 1, \ldots, N-1$. The symbol $\prod$ is used for the ordered product

$$\prod_{j=1}^{N-1} P_j = P_1 \cdot P_2 \cdot \cdots \cdot P_{N-1} \tag{2.7}$$

The translation operator $T(\alpha, N)$ is related to the momentum operator $P(\alpha, N)$ by

$$T(\alpha, N) = \exp[-i \cdot P(\alpha, N)] \tag{2.8}$$

Note that $T(\alpha, N)$ is a unitary operator on $\mathscr{H}(N)$ [and hence $P(\alpha, N)$ is Hermitian] if $|\alpha| = 1$ and that $T(\alpha, N)^N = \alpha^{-S^z}$ and therefore is constant on each subspace $\mathscr{H}_Q(N)$, (2.3).

For later convenience we introduce a "charge conjugation" operator C defined by

$$C = \prod_{j=1}^{N} \sigma_j^x = C^\dagger = C^{-1} \tag{2.9}$$

It satisfies the relations $C \cdot H(q, \alpha, N) \cdot C = H(q, \alpha^{-1}, N)$, $C \cdot T(\alpha, N) \cdot C = T(\alpha^{-1}, N)$, and $C \cdot S^z \cdot C = -S^z$.

We are now going to present an extended version of the projection mechanism developed in ref. 1. For this purpose let us recall the universal finite-size scaling limit partition function of the spin-1/2 *XXZ* Heisenberg chain. In what follows, we restrict ourselves to $|q| = |\alpha| = 1$ in Eq. (2.1), which ensures the hermiticity of both the Hamiltonian $H(q, \alpha, N)$ of (2.1) and the momentum operator $P(\alpha, N)$ in (2.8). We use the parametrization

$$q = -\exp(-i\pi\gamma), \qquad \alpha = \exp(2\pi i l) \tag{2.10}$$

with two real numbers $0 \leqslant \gamma < 1$ and $-1/2 < l \leqslant 1/2$. The Hamiltonian (2.1) equipped with the normalization factor that guarantees an isotropic continuum limit is given by

$$H^l(h, N) = \frac{\gamma}{2\sin(\pi\gamma)} H(q = -\exp(-i\pi\gamma), \alpha = \exp(2\pi i l), N) \tag{2.11}$$

where $h \geqslant 1/4$ is given by

$$h = \tfrac{1}{4}(1-\gamma)^{-1} \tag{2.12}$$

The finite-size scaling limit of this system is known to be described by the $c = 1$ conformal field theory of a free compactified boson with the compactification radius $\mathscr{R}$ being related to the anisotropy γ by $\mathscr{R}^2 = 8h$ (see, e.g., ref. 14). Denoting the eigenvalues of $H^l_Q(h, N) = H^l(h, N) \cdot \mathscr{P}_Q$ in the charge sector Q with $-N/2 \leqslant Q \leqslant N/2$ by $E^l_{Q;j}(h, N)$ and the corresponding momenta by

$$P^l_{Q;j}(h, N), \qquad j = 1, 2, \ldots, \binom{N}{Q + N/2}$$

one obtains the following expression for the finite-size scaling partition function of $H^l_Q(h, N)$[10,15]:

$$\begin{aligned}
\mathscr{E}^l_Q(z, \bar{z}) &= \lim_{N\to\infty} \mathscr{E}^l_Q(z, \bar{z}, N) \\
&= \lim_{N\to\infty} \sum_{j=1}^{\binom{N}{Q+N/2}} z^{(1/2)(\bar{E}^l_{Q;j}(h,N) + \bar{P}^l_{Q;j}(h,N))} \bar{z}^{(1/2)(\bar{E}^l_{Q;j}(h,N) - \bar{P}^l_{Q;j}(h,N))} \\
&= \sum_{\nu\in\mathbb{Z}} z^{[Q+4h(l+\nu)]^2/16h} \Pi_V(z)\, \bar{z}^{[Q-4h(l+\nu)]^2/16h} \Pi_V(\bar{z})
\end{aligned} \tag{2.13}$$

In this equation, $\bar{E}^{l}_{Q;j}(h, N)$ and $\bar{P}^{l}_{Q;j}(h, N)$ denote the scaled gaps[16]

$$\bar{E}^{l}_{Q;j}(h, N) = \frac{N}{2\pi}\,(E^{l}_{Q;j}(h, N) - E_0(h, N)) \tag{2.14}$$

$$\bar{P}^{l}_{Q;j}(h, N) = \frac{N}{2\pi}\,P^{l}_{Q;j}(h, N) \tag{2.15}$$

where $E_0(h, N) = E^{0}_{0;1}(h, N)$ is the ground-state energy of the periodic Hamiltonian and $\Pi_V(z) = \prod_{n=1}^{\infty}(1 - z^n)^{-1}$ is the generating function of the number of partitions. Equation (2.15) has to be understood carefully. Since the finite-size scaling partition function (2.13) should only contain the universal term of the partition function, one has to neglect macroscopic momenta (i.e., momenta of order one). To be more precise, the levels that contribute to the partition function $\mathscr{E}^{l}_{Q}(z, \bar{z})$ in Eq. (2.13) have a macroscopic momentum of modulus 0 or π, depending on ν being even or odd, i.e., Eq. (2.15) should be modified as follows:

$$\bar{P}^{l}_{Q;j}(h, N) = \frac{N}{2\pi}\,[P^{l}_{Q;j}(h, N) - \pi\kappa^{l}_{Q;j}(N)] \tag{2.16}$$

where $\kappa^{l}_{Q;j}(N) \in \{0, 1\}$, depending on the value of the macroscopic momentum of the corresponding level. Note that the scaled momenta are defined modulo N (since the momenta are defined modulo 2π).

It is now our aim to extract the $c < 1$ character function of irreducible representations of the Virasoro algebra out of the partition functions $\mathscr{E}^{l}_{Q}(z, \bar{z})$ of (2.13). We parametrize the central charge $c < 1$ (we consider only real values of c) by a positive real number m by

$$c = 1 - \frac{6}{m(m+1)} \tag{2.17}$$

Then one has to distinguish between the case that m is a rational number (corresponding to minimal models) and the other case that m is irrational (corresponding to nonminimal models, i.e., the number of irreducible highest weight representations is infinite in this case). For *irrational* values of m, the character functions $\chi_{r,s}(z) = \mathrm{tr}(z^{L_0})$ (L_0 generates, besides the central element c, the Cartan subalgebra of the Virasoro algebra) for a highest weight representation with highest weight $\Delta_{r,s}$,

$$\Delta_{r,s} = \frac{[(m+1)\,r - ms]^2 - 1}{4m(m+1)} \tag{2.18}$$

with $r, s = 1, 2,\ldots$, are given by

$$\chi_{r,s}(z) = (z^{\Delta_{r,s}} - z^{-\Delta_{r,s}})\, \Pi_V(z) \tag{2.19}$$

If, however, m is *rational*, say $m = u/v$ with coprime positive integers u and v, the characters are[17]

$$\chi_{r,s}(z) = \Omega_{r,s}(z) - \Omega_{r,-s}(z) \tag{2.20}$$

$$\Omega_{r,s}(z) = \sum_{\nu \in \mathbb{Z}} z^{\{[2u(u+v)\nu + (u+v)r - us]^2 - v^2\}/4u(u+v)} \Pi_V(z) \tag{2.21}$$

and we can restrict the possible values of r and s to the set

$$1 \leqslant r \leqslant u-1, \qquad 1 \leqslant s \leqslant u+v-1 \tag{2.22}$$

The representations are unitary for integer values of m,[18] i.e., for $v = 1$.

The shift of the central charge c from the free boson value $c = 1$ to a value $c < 1$ [see (2.17)] is now performed by choosing a new ground state. For this let us use the level $E^{l_0}_{0;j_0}(h, N)$ in the charge sector $Q = 0$ (note that we do not necessarily take the lowest eigenvalue in this sector) of the Hamiltonian (2.11) with boundary condition l_0. The number $j_0 \geqslant 1$ has to be chosen such that this level corresponds to the one that gives the contribution $z^{h(l_0+\nu_0)^2}\bar{z}^{h(l_0+\nu_0)^2}$ in the partition function (2.13), where $(l_0 + \nu_0)^2$ is related to h by

$$c = 1 - \frac{6}{m(m+1)} = 1 - 24h(l_0 + \nu_0)^2 \tag{2.23}$$

or equivalently to γ by $\gamma = 1 - m(m+1)(l_0 + \nu_0)^2$. (This means that j_0 labels the level that contributes the difference of the universal parts of the ground-state energies $z^{(1-c)/24}\bar{z}^{(1-c)/24}$ between the original XXZ Heisenberg chain with $c = 1$ and the system we wish to project out.) Here, $-1/2 \leqslant l_0 \leqslant 1/2$ and ν_0 is an integer. For definiteness, we choose the square root in Eq. (2.23) to be positive, hence $l_0 + \nu_0 = [4hm(m+1)]^{-1/2}$.

We now define *new* scaled gaps with respect to our new ground state $E^{l_0}_{0;j_0}(h, N)$ with h satisfying Eq. (2.23) by [cf. Eqs. (2.14)–(2.16)]

$$\bar{F}^k_{Q;j}(N) = \frac{N}{2\pi} [E^{k(l_0+\nu_0)}_{Q;j}(h, N) - E^{l_0}_{0;j_0}(h, N)] \tag{2.24}$$

$$\bar{P}^k_{Q;j}(N) = \frac{N}{2\pi} [P^{k(l_0+\nu_0)}_{Q;j}(h, N) - \pi\tilde{\kappa}^k_{Q;j}(N)] \tag{2.25}$$

where k for the moment is arbitrary and

$$\tilde{\kappa}^k_{Q;j} = \kappa^{k(l_0+\nu_0)}_{Q;j} - \kappa^{l_0+\nu_0}_{0;j_0}$$

Note that the momentum $P^{l_0}_{0;j_0}(h, N)$ of the new ground state always vanishes [again this is only true up to a possible shift of π; see the remark concerning Eqs. (2.15) and (2.16)]. The corresponding finite-size scaling partition function is given by [cf. Eq. (2.13)]

$$\begin{aligned}\mathscr{F}^k_Q(z, \bar{z}) &= \lim_{N \to \infty} \mathscr{F}^k_Q(z, \bar{z}, N) \\ &= \lim_{N \to \infty} \sum_{j=1}^{\binom{N}{Q+N/2}} z^{(1/2)(\bar{F}^k_{Q;j}(N) + \bar{P}^k_{Q;j}(N))} \bar{z}^{(1/2)(\bar{F}^k_{Q;j}(N) - \bar{P}^k_{Q;j}(N))} \\ &= \sum_{\nu \in \mathbb{Z}} z^{\{[m(m+1)(l_0+v_0)Q + k + \nu/(l_0+v_0)]^2 - 1\}/4m(m+1)} \Pi_V(z) \\ &\quad \times z^{\{[m(m+1)(l_0+v_0)Q - k - \nu/(l_0+v_0]^2 - 1\}/4m(m+1)} \Pi_V(\bar{z}) \end{aligned} \tag{2.26}$$

We are now in a position to obtain the finite-size scaling partition function of the unitary and nonunitary models with central charge less than one. Equation (2.23) gives c as a function of the two free (real) parameters h and $l_0 + v_0$. Following closely the results of ref. 1, we define two classes of series of $c < 1$ models by relating h and $l_0 + v_0$ through

$$l_0 + v_0 = \frac{1}{M} - \frac{M}{4h} \tag{2.27}$$

which we shall call the R-models if $M > 0$ and the L-models if $M < 0$. We label these two series of models by the positive number $R = M$ or $L = -M$, respectively ($M \neq 0$), which will be integer throughout this paper.[3] We first investigate the R-models ($M = R > 0$).

2.1. The *R*-models

According to Eq. (2.27), we define the R-models by the following relation between h and m, which defines the central charge according to Eq. (2.17):

$$h = \frac{R^2}{4} \frac{m+1}{m} \tag{2.28}$$

which means that $h > R^2/4$ for all m. Using Eqs. (2.12) and (2.23), one obtains

$$\gamma = 1 - \frac{m}{R^2(m+1)}, \qquad l_0 + v_0 = \frac{1}{R(m+1)} \tag{2.29}$$

where we shall consider only integer values of R.

[3] Note that we change the notation compared to ref. 1 at this point.

For *m irrational,* the finite-size scaling partition functions $\mathscr{F}^k_Q(z, \bar{z})$ of (2.26) are given by [see Eq. (2.18)]

$$\mathscr{F}^k_Q(z, \bar{z}) = \sum_{v \in \mathbb{Z}} z^{\Delta_{vR+k,k-Q/R}} \Pi_V(z)\, \bar{z}^{\Delta_{vR+k,k+Q/R}} \Pi_V(\bar{z}) \tag{2.30}$$

One recognizes that for integer values of R one can recover the character functions (2.19) through

$$\begin{aligned} D^{Rf}_{R^2g}(z, \bar{z}) &= \mathscr{F}^{Rf}_{R^2g}(z, \bar{z}) - \mathscr{F}^{Rg}_{R^2f}(z, \bar{z}) \\ &= \sum_{r=1}^{\infty} \chi_{Rr, R(f-g)}(z)\, \chi_{Rr, R(f+g)}(\bar{z}) \end{aligned} \tag{2.31}$$

where f and g are integers with $f > 0$ and $-f < g < f$. These quantities are the finite-size scaling partition functions of nonminimal models with central charge given by (2.17). The integer numbers R^2g and Rf label the boundary conditions and sectors of the projected systems according their internal global symmetries, $g = 0$ corresponding to periodic boundary conditions. Note that a vacuum representation $(\chi_{1,1}(z)\, \chi_{1,1}(\bar{z}))$ only occurs for the case $R = 1$ (corresponding to the 2_R models of ref. 1). The same is true for the modular invariant partition function

$$\mathscr{A}_m(z, \bar{z}) = \sum_{r,s=1}^{\infty} \chi_{r,s}(z)\, \chi_{r,s}(\bar{z}) \tag{2.32}$$

which for $R = 1$ can be written as the sum of the sectors D^f_0 of Eq. (2.31),

$$\mathscr{A}_m(z, \bar{z}) = \sum_{f=1}^{\infty} D^f_0(z, \bar{z}) \tag{2.33}$$

which is impossible for $R \neq 1$.

We now turn to the case *m rational,* $m = u/v$, with coprime positive integers u and v. Equations (2.28) and (2.29) take the form

$$h = \frac{R^2}{4} \frac{u+v}{u}, \quad \gamma = 1 - \frac{u}{R^2(u+v)}, \quad l_0 + v_0 = \frac{v}{R(u+v)} \tag{2.34}$$

The finite-size scaling partition function $\mathscr{F}^k_Q(z, \bar{z})$ now has the periodicity property $\mathscr{F}^k_Q(z, \bar{z}) = \mathscr{F}^{k \pm n}_Q(z, \bar{z})$ with the integer n being defined by

$$n = R(u+v) \tag{2.35}$$

and $l_0 + v_0 = v/n$. From ref. 1 we know that instead of the partition functions $\mathscr{F}^k_Q(z, \bar{z})$ we should consider

$$\begin{aligned}\mathscr{G}^k_Q(z,\bar z)&=\sum_{\mu\in\mathbb{Z}}\mathscr{F}^k_{Q+n\mu}(z,\bar z)\\&=\sum_{\mu,\nu\in\mathbb{Z}} z^{\{[u(u+v)(Q/n+\mu)+kv+nv]^2-v^2\}/4u(u+v)}\Pi_V(z)\\&\quad\times\bar z^{\{[u(u+v)(Q/n+\mu)-kv-nv]^2-v^2\}/4u(u+v)}\Pi_V(\bar z)\\&=\mathscr{G}^k_{Q\pm n}(z,\bar z)=\mathscr{G}^{k\pm n}_Q(z,\bar z)=\mathscr{G}^{n-k}_{n-Q}(z,\bar z)\end{aligned}\tag{2.36}$$

that is, we sum over all charge sectors modulo n, (2.35). If $u(u+v)/n=u/R$ is integer (i.e., u is a multiple of R), one can rewrite this expression as follows:

$$\begin{aligned}\mathscr{G}^k_Q(z,\bar z)&=\sum_{w=0}^{2u/R-1}\left\{\sum_{\mu\in\mathbb{Z}} z^{\{[2u(u+v)\mu+u(Q/R)+kv+(u+v)Rw]^2-v^2\}/4u(u+v)}\Pi_V(z)\right\}\\&\quad\times\left\{\sum_{\nu\in\mathbb{Z}} \bar z^{\{[2u(u+v)\nu+u(Q/R)-kv-(u+v)Rw]^2-v^2\}/4u(u+v)}\Pi_V(\bar z)\right\}\\&=\sum_{w=0}^{2u/R-1}\Omega_{Rw+k,k-Q/R}(z)\,\Omega_{Rw+k,k+Q/R}(\bar z)\end{aligned}\tag{2.37}$$

Comparing this with Eq. (2.30), one realizes that there are again differences of these partition functions that can be written as bilinear expressions in the characters (2.20)

$$\begin{aligned}D^{Rf}_{R^2g}(z,\bar z)&=D^{R(u+v-f)}_{R(u+v-Rg)}(z,\bar z)\\&=\mathscr{G}^{Rf}_{R^2g}(z,\bar z)-\mathscr{G}^{Rg}_{R^2f}(z,\bar z)\\&=\sum_{w=1}^{u/R-1}\chi_{Rw,R(f-g)}(z)\,\chi_{Rw,R(f+g)}(\bar z)\end{aligned}\tag{2.38}$$

where now f and g are integers that satisfy the inequalities $1\leqslant Rf\leqslant u+v-1$ and $|Rg|\leqslant\min\{Rf-1,u+v-1-Rf\}$. Here, the quantities are the finite-size scaling partition functions of minimal models with central charge $c=1-6v^2/[u(u+v)]$ in the sector Rf with boundary condition R^2g, with $g=0$ corresponding to periodic boundary conditions in the projected system. The unitary series is given by $v=1$. Note that, as discussed above, c does not depend on the choice of R. As will be shown below, different values for R lead to different systems with the same central charge.

In addition to these sectors there are sectors that occur only for certain rational values of m and partly also include the half-integer charge sectors of the Hamiltonian (2.11). In what follows, we limit our discussion of these possibilities to the $R=1$ and $R=2$ models (corresponding to the

2_R and 1_R models of ref. 1, respectively). Let us start exploring the $R=1$ models.

2.1.1. The $R=1$ Models. The unitary subset (i.e., $v=1$) of the $R=1$ series was discussed in ref. 1, where it was called the 2_R-series. It represents the p-state Potts models with $p=\{2\cos[\pi/(u+1)]\}^2$. With $n=u+v$ [see Eq. (2.35)] for general v, Eq. (2.38) just becomes

$$\begin{aligned} D^f_g(z,\bar{z}) = D^{n-f}_{n-g}(z,\bar{z}) &= \mathcal{G}^f_g(z,\bar{z}) - \mathcal{G}^g_f(z,\bar{z}) \\ &= \sum_{r=1}^{u-1} \chi_{r,f-g}(z)\,\chi_{r,f+g}(\bar{z}) \end{aligned} \tag{2.39}$$

for all values of u and

$$1 \leqslant f \leqslant n-1, \qquad |g| \leqslant \min\{f-1, n-1-f\} \tag{2.40}$$

These sectors (involving only integer charge sectors of the XXZ Heisenberg chain) have the same structure as those obtained in ref. 1.

By a closer inspection of Eq. (2.37) one realizes that one has the following possibilities to use half-integer values for k and Q in the partition function $\mathcal{G}^k_Q(z,\bar{z})$ of (2.36): For <u>u even</u>, k has to be integer and Q may be either integer or half-integer, for <u>$u+v$ even</u>, k and Q have both to be integer or half-integer numbers, whereas for the case <u>v even</u> only k may be half-integer-valued and Q has to be an integer. Defining $\tau=[1-(-1)^{2\tilde{g}}]/4$, one obtains as a generalization of Eq. (2.39)

$$\begin{aligned} D^{\tilde{f}}_{\tilde{g}}(z,\bar{z}) = D^{n-\tilde{f}}_{n-\tilde{g}}(z,\bar{z}) &= \mathcal{G}^{\tilde{f}}_{\tilde{g}}(z,\bar{z}) - \mathcal{G}^{\tilde{g}+\tau n}_{\tilde{f}+\tau n} \\ &= \sum_{r=1}^{u-1} \chi_{r,\tilde{f}-\tilde{g}+\tau n}(z)\,\chi_{r,\tilde{f}+\tilde{g}+\tau n}(\bar{z}) \end{aligned} \tag{2.41}$$

where now $\tilde{f}$ and $\tilde{g}$ take the values specified above and have to be chosen such that the conditions (2.22) are fulfilled. To decide if in this way we really get *new* sectors [remember that one has to fulfill the conditions (2.22)], we have a closer look at the sectors appearing in Eq. (2.41), beginning with the case u even.

a. <u>*u Even.*</u> Here one can rewrite the sectors of Eq. (2.41) involving half-integer charges as

$$\begin{aligned} D^g_{f-n/2}(z,\bar{z}) = D^{n-g}_{n/2-f}(z,\bar{z}) &= \mathcal{G}^g_{f-n/2}(z,\bar{z}) - \mathcal{G}^f_{g+n/2}(z,\bar{z}) \\ &= \sum_{r=1}^{u-1} \chi_{u-r,f-g}(z)\,\chi_{r,f+g}(\bar{z}) \end{aligned} \tag{2.42}$$

where now f and g are integers again. Clearly these sectors fulfill the conditions (2.22) if f and g satisfy Eq. (2.40) and one obtains *new* sectors in this way as long as u is not equal to two. In this special case the sectors nontrivially coincide and one obtains the same characters from an even and an odd number of sites. This will be reflected by special features of the finite-size spectra (see the discussion in Section 4.2 below), as already observed for the free chain case (see ref. 6). As an example, let us consider the simplest model of this kind, which is the nonunitary $u=2$, $v=3$ model corresponding to a central charge of $c=-22/5$. The sectors of Eqs. (2.39) and (2.42) are

$$\begin{aligned}
D_0^1 = D_0^4 = \mathcal{G}_0^1 - \mathcal{G}_1^0 &= (0, 0) &&= \mathcal{G}_{3/2}^0 - \mathcal{G}_{5/2}^4 = D_{7/2}^0 = D_{3/2}^0 \\
D_0^2 = D_0^3 = \mathcal{G}_0^2 - \mathcal{G}_2^0 &= (-\tfrac{1}{5}, -\tfrac{1}{5}) &&= \mathcal{G}_{1/2}^0 - \mathcal{G}_{5/2}^3 = D_{9/2}^0 = D_{1/2}^0 \\
D_1^2 = D_4^3 = \mathcal{G}_1^2 - \mathcal{G}_2^1 &= (0, -\tfrac{1}{5}) &&= \mathcal{G}_{1/2}^4 - \mathcal{G}_{3/2}^3 = D_{9/2}^1 = D_{1/2}^4 \\
D_1^3 = D_4^2 = \mathcal{G}_1^3 - \mathcal{G}_3^1 &= (-\tfrac{1}{5}, 0) &&= \mathcal{G}_{1/2}^1 - \mathcal{G}_{7/2}^3 = D_{9/2}^4 = D_{1/2}^1
\end{aligned} \tag{2.43}$$

where we used the notation $(\Delta_{r,s}, \Delta_{r',s'})$ for the product of characters $\chi_{r,s}(z)\,\chi_{r',s'}(\bar{z})$ according to the highest weights of the corresponding irreducible representations of the Virasoro algebra. As an example for the other possibility, $u \neq 2$, let us look at the unitary model $u=4$, $v=1$ with a central charge $c=7/10$. Since $n=u+v=5$ as in the example above, the structure of the sectors is exactly the same, but one obtains

$$\begin{aligned}
D_0^1 = D_0^4 = \mathcal{G}_0^1 - \mathcal{G}_1^0 &= (0, 0) + (\tfrac{7}{16}, \tfrac{7}{16}) + (\tfrac{3}{2}, \tfrac{3}{2}) \\
D_0^2 = D_0^3 = \mathcal{G}_0^2 - \mathcal{G}_2^0 &= (\tfrac{1}{10}, \tfrac{1}{10}) + (\tfrac{3}{80}, \tfrac{3}{80}) + (\tfrac{3}{5}, \tfrac{3}{5}) \\
D_1^2 = D_4^3 = \mathcal{G}_1^2 - \mathcal{G}_2^1 &= (0, \tfrac{3}{5}) + (\tfrac{7}{16}, \tfrac{3}{80}) + (\tfrac{3}{2}, \tfrac{1}{10}) \\
D_1^3 = D_4^2 = \mathcal{G}_1^3 - \mathcal{G}_3^1 &= (\tfrac{1}{10}, \tfrac{3}{2}) + (\tfrac{3}{80}, \tfrac{7}{16}) + (\tfrac{3}{5}, 0) \\
D_{3/2}^0 = D_{7/2}^0 = \mathcal{G}_{3/2}^0 - \mathcal{G}_{5/2}^4 &= (\tfrac{3}{2}, 0) + (\tfrac{7}{16}, \tfrac{7}{16}) + (0, \tfrac{3}{2}) \\
D_{1/2}^0 = D_{9/2}^0 = \mathcal{G}_{1/2}^0 - \mathcal{G}_{5/2}^3 &= (\tfrac{3}{5}, \tfrac{1}{10}) + (\tfrac{3}{80}, \tfrac{3}{80}) + (\tfrac{1}{10}, \tfrac{3}{5}) \\
D_{1/2}^4 = D_{9/2}^1 = \mathcal{G}_{1/2}^4 - \mathcal{G}_{3/2}^3 &= (0, \tfrac{1}{10}) + (\tfrac{7}{16}, \tfrac{3}{80}) + (\tfrac{3}{2}, \tfrac{3}{5}) \\
D_{1/2}^1 = D_{9/2}^4 = \mathcal{G}_{1/2}^1 - \mathcal{G}_{7/2}^3 &= (\tfrac{1}{10}, 0) + (\tfrac{3}{80}, \tfrac{7}{16}) + (\tfrac{3}{5}, \tfrac{3}{2})
\end{aligned} \tag{2.44}$$

The half-integer charge sectors do not coincide with any integer charge sector.

b. <u>*u + v Even.*</u> Here we consider the case that both $\tilde{f}$ and $\tilde{g}$ in Eq. (2.41) are half-integer numbers. We obtain

$$\begin{aligned}
D_{\tilde{g}}^{\tilde{f}-n/2}(z, \bar{z}) = D_{n-\tilde{g}}^{n/2-\tilde{f}}(z, \bar{z}) &= \mathcal{G}_{\tilde{g}}^{\tilde{f}-n/2} - \mathcal{G}_{\tilde{f}}^{\tilde{g}+n/2} \\
&= \sum_{r=1}^{u-1} \chi_{r,\tilde{f}-\tilde{g}}(z)\,\chi_{r,\tilde{f}+\tilde{g}}(\bar{z})
\end{aligned} \tag{2.45}$$

and therewith *new* sectors if the half-integer numbers $\tilde{f}$ and $\tilde{g}$ satisfy the same relations as the integers f and g in Eq. (2.40). Here we take the $u=3$, $v=1$ model, which corresponds to the Ising model with central charge $c=1/2$, and the $u=5$, $v=1$ model, which corresponds to the 3-state Potts model with central charge $c=4/5$,[1] as two examples. From Eqs. (2.39) and (2.45) one obtains the following sectors for $u=3$:

$$\begin{aligned}
D_0^1 = D_0^3 &= \mathscr{G}_0^1 - \mathscr{G}_1^0 = (0,0) + (\tfrac{1}{2},\tfrac{1}{2}) \\
D_0^2 &= \mathscr{G}_0^2 - \mathscr{G}_2^0 = 2\cdot(\tfrac{1}{16},\tfrac{1}{16}) \\
D_1^2 = D_3^2 &= \mathscr{G}_1^2 - \mathscr{G}_2^1 = (0,\tfrac{1}{2}) + (\tfrac{1}{2},0) \\
D_{1/2}^{1/2} = D_{7/2}^{7/2} &= \mathscr{G}_{1/2}^{1/2} - \mathscr{G}_{5/2}^{5/2} = (\tfrac{1}{16},0) + (\tfrac{1}{16},\tfrac{1}{2}) \\
D_{1/2}^{7/2} = D_{7/2}^{1/2} &= \mathscr{G}_{1/2}^{7/2} - \mathscr{G}_{3/2}^{5/2} = (0,\tfrac{1}{16}) + (\tfrac{1}{2},\tfrac{1}{16})
\end{aligned} \tag{2.46}$$

and for $u=5$ one has

$$\begin{aligned}
D_0^1 = D_0^5 &= \mathscr{G}_0^1 - \mathscr{G}_1^0 = (0,0) + (\tfrac{2}{5},\tfrac{2}{5}) + (\tfrac{7}{5},\tfrac{7}{5}) + (3,3) \\
D_0^2 = D_0^4 &= \mathscr{G}_0^2 - \mathscr{G}_2^0 = (\tfrac{1}{8},\tfrac{1}{8}) + (\tfrac{1}{40},\tfrac{1}{40}) + (\tfrac{21}{40},\tfrac{21}{40}) + (\tfrac{13}{8},\tfrac{13}{8}) \\
D_0^3 &= \mathscr{G}_0^3 - \mathscr{G}_3^0 = 2\cdot(\tfrac{2}{3},\tfrac{2}{3}) + 2\cdot(\tfrac{1}{15},\tfrac{1}{15}) \\
D_1^2 = D_5^4 &= \mathscr{G}_1^2 - \mathscr{G}_2^1 = (0,\tfrac{2}{3}) + (\tfrac{2}{5},\tfrac{1}{15}) + (\tfrac{7}{5},\tfrac{1}{15}) + (3,\tfrac{2}{3}) \\
D_1^3 = D_5^3 &= \mathscr{G}_1^3 - \mathscr{G}_3^1 = (\tfrac{1}{8},\tfrac{13}{8}) + (\tfrac{1}{40},\tfrac{21}{40}) + (\tfrac{21}{40},\tfrac{1}{40}) + (\tfrac{13}{8},\tfrac{1}{8}) \\
D_1^4 = D_5^2 &= \mathscr{G}_1^4 - \mathscr{G}_4^1 = (\tfrac{2}{3},0) + (\tfrac{1}{15},\tfrac{2}{5}) + (\tfrac{1}{15},\tfrac{7}{5}) + (\tfrac{2}{3},3) \\
D_2^3 = D_4^3 &= \mathscr{G}_2^3 - \mathscr{G}_3^2 = (0,3) + (\tfrac{2}{5},\tfrac{7}{5}) + (\tfrac{7}{5},\tfrac{2}{5}) + (3,0) \\
D_{1/2}^{1/2} = D_{11/2}^{11/2} &= \mathscr{G}_{1/2}^{1/2} - \mathscr{G}_{7/2}^{7/2} = (\tfrac{2}{3},\tfrac{1}{8}) + (\tfrac{1}{15},\tfrac{1}{40}) + (\tfrac{1}{15},\tfrac{21}{40}) + (\tfrac{2}{3},\tfrac{13}{8}) \\
D_{1/2}^{3/2} = D_{11/2}^{9/2} &= \mathscr{G}_{1/2}^{3/2} - \mathscr{G}_{9/2}^{7/2} = (\tfrac{1}{8},0) + (\tfrac{1}{40},\tfrac{2}{5}) + (\tfrac{21}{40},\tfrac{7}{5}) + (\tfrac{13}{8},3) \\
D_{1/2}^{9/2} = D_{11/2}^{3/2} &= \mathscr{G}_{1/2}^{9/2} - \mathscr{G}_{3/2}^{7/2} = (0,\tfrac{1}{8}) + (\tfrac{2}{5},\tfrac{1}{40}) + (\tfrac{7}{5},\tfrac{21}{40}) + (3,\tfrac{13}{8}) \\
D_{1/2}^{11/2} = D_{11/2}^{1/2} &= \mathscr{G}_{1/2}^{11/2} - \mathscr{G}_{5/2}^{7/2} = (\tfrac{1}{8},\tfrac{2}{3}) + (\tfrac{1}{40},\tfrac{1}{15}) + (\tfrac{21}{40},\tfrac{1}{15}) + (\tfrac{13}{8},\tfrac{2}{3}) \\
D_{3/2}^{1/2} = D_{9/2}^{11/2} &= \mathscr{G}_{3/2}^{1/2} - \mathscr{G}_{7/2}^{9/2} = (\tfrac{13}{8},0) + (\tfrac{21}{40},\tfrac{2}{5}) + (\tfrac{1}{40},\tfrac{7}{5}) + (\tfrac{1}{8},3) \\
D_{3/2}^{11/2} = D_{9/2}^{1/2} &= \mathscr{G}_{3/2}^{11/2} - \mathscr{G}_{5/2}^{9/2} = (0,\tfrac{13}{8}) + (\tfrac{2}{5},\tfrac{21}{40}) + (\tfrac{7}{5},\tfrac{1}{40}) + (3,\tfrac{1}{8})
\end{aligned} \tag{2.47}$$

Here, not only are the half-integer sectors as such new (i.e., the *combination* of the various building blocks contributing to it), in addition they are built by contributions from so far unknown spinor fields with anomalous dimensions $(\Delta, \bar{\Delta})$ as given in (2.46) and (2.47). The partition functions of the form $D_0^{n/2}$ are special insofar as the sector in the *XXZ* Heisenberg chain from which they are obtained splits into two subsectors with eigenvalue

$C = \pm 1$ of the charge conjugation operator C, (2.9). This symmetry is not used in the projection mechanism and is presumably the reason why these partition functions contain each contribution twice.[4] We will return to these models later when we study the corresponding finite systems.

c. *v Even.* In this case one would expect new sectors appearing for half-integer values of $\tilde{f}$ in Eq. (2.41), but this is not the case, due to the identity $\mathscr{G}_Q^{k \pm n/2}(z, \bar{z}) \equiv \mathscr{G}_Q^k(z, \bar{z})$, which follows simply from the fact that $(n/2)(l_0 + v_0) = v/2$ is an integer.

2.1.2. The $R = 2$ Models. Here the unitary subset $v = 1$ discussed in ref. 1 (called 1_R models there) corresponds to the low-temperature $O(p)$ models[19,20] with $p = 2\cos[\pi/(u+1)]$. We have to investigate u even and u odd separately (u odd was not discussed in ref. 1), as we can use Eq. (2.37) for *u even* only, i.e., for $n = 2(u+v) \equiv 2 \bmod 4$ (since v has to be odd). Now Eq. (2.38) reads

$$\begin{aligned} D_{4g}^{2f}(z, \bar{z}) = D_{n-4g}^{n-2f}(z, \bar{z}) &= \mathscr{G}_{4g}^{2f}(z, \bar{z}) - \mathscr{G}_{4f}^{2g}(z, \bar{z}) \\ &= \sum_{w=1}^{u/2-1} \chi_{2w, 2(f-g)}(z)\, \chi_{2w, 2(f+g)}(\bar{z}) \end{aligned} \tag{2.48}$$

There are, however, additional sectors besides these. But only integer charge sectors contribute here, since, although one can form bilinear expressions in character functions from the partition functions of the half-integer charge sectors, these involve negative multiplicities, which we do not want to consider in our present discussion. For completeness, we will briefly state the relevant equations.

a. *u Even.* One obtains the following expression for the $\mathscr{G}_Q^k(z, \bar{z})$ of (2.37) in terms of the functions $\Omega_{r,s}(z)$ of (2.21):

$$\begin{aligned} \mathscr{G}_Q^k(z, \bar{z}) = &\sum_{d=1-\xi}^{u/2-\xi} \Omega_{\xi+2d, k-Q/2+\eta(u+v)} \Omega_{\xi+2d, k+Q/2+\eta(u+v)}(\bar{z}) \\ &+ \sum_{d=0}^{u/2-1} \Omega_{-(\xi+2d), k-Q/2+\eta(u+v)}(z)\, \Omega_{-(\xi+2d), k+Q/2+\eta(u+v)}(\bar{z}) \end{aligned} \tag{2.49}$$

where $\eta = [1 - (-1)^Q]/4$ and $\xi = [1 - (-1)^k]/2$ $\{\xi = [1 - (-1)^{k+Q}]/2\}$ if $u \equiv 0 \bmod 4$ ($u \equiv 2 \bmod 4$), respectively. From this, one gets in generalization of Eq. (2.48) the sectors

$$D_{4g+t}^{2f+\omega}(z, \bar{z}) = \mathscr{G}_{4g+t}^{2f+\omega}(z, \bar{z}) - \mathscr{G}_{2(2f+\omega)+t(u+v)}^{(4g+t)/2+\varepsilon((2\omega+t)/2)(u+v)}(z, \bar{z}) \tag{2.50}$$

[4] We did not check this assumption.

with $\omega \in \{0, 1\}$, $t \in \{0, 1, 2, 3\}$, and $\varepsilon = (-1)^{(u+v+1)/2}$. In terms of character functions, they have the form

$$D_Q^k(z, \bar{z}) = \sum_{d=1-\xi}^{u/2-1} \chi_{\xi+2d, k-Q/2+\eta(u+v)}(z)\, \chi_{\xi+2d, k+Q/2+\eta(u+v)}(\bar{z}) \tag{2.51}$$

For physical sectors, of course, the character functions which enter on the right-hand side of Eq. (2.51) have to comply with the conditions (2.22), taking into account the periodicity properties of the character functions $\chi_{r,s}(z, \bar{z})$ in the indices r and s.

b. u Odd. Although, as mentioned above, Eq. (2.37) does not apply for odd values of u, one can construct $R = 2$ models in this case, too. However, one has to consider new partition functions $\tilde{\mathscr{G}}_Q^k(z, \bar{z})$ which are the sums of two sectors $\mathscr{G}_Q^k$,

$$\tilde{\mathscr{G}}_Q^k(z, \bar{z}) = \mathscr{G}_Q^k(z, \bar{z}) + \mathscr{G}_Q^{k+n/2}(z, \bar{z}) \tag{2.52}$$

in order to obtain suitable expressions. Following the same procedure which led to Eq. (2.37) yields

$$\begin{aligned} \tilde{\mathscr{G}}_Q^k(z, \bar{z}) &= \sum_{\mu, v \in \mathbb{Z}} \{ z^{\{[u(u+v)\mu + (u/2)Q + kv + (u+v)v]^2 - v^2\}/4u(u+v)} \Pi_V(z) \\ &\quad \times \bar{z}^{\{[u(u+v)\mu + (u/2)Q - kv - (u+v)v]^2 - v^2\}/4u(u+v)} \Pi_V(\bar{z}) \} \\ &= \sum_{w=0}^{u-1} \Omega_{w+k, k-Q/2}(z)\, \Omega_{w+k, k+Q/2}(\bar{z}) \end{aligned} \tag{2.53}$$

Hence, the new partition functions $\tilde{\mathscr{G}}_Q^k(z, \bar{z})$ are in fact the same as the sectors $\mathscr{G}_{Q/2}^k(z, \bar{z})$ [see Eq. (2.37)] of the corresponding $R = 1$ model with $n = u + v$. Of course all the equations obtained there translate to the present case.

2.2. The L-Models

Similar to the R-models, we define the L-models by

$$h = \frac{L^2}{4} \frac{m}{m+1}, \qquad \gamma = 1 - \frac{m+1}{L^2 m}, \qquad l_0 + v_0 = \frac{1}{Lm} \tag{2.54}$$

where in what follows L will be integer-valued again and $h < L^2/4$ for all possible values of m.

Let us commence by considering *irrational* values of m. The finite-size scaling partition functions $\mathscr{F}^k_Q(z,\bar{z})$ of (2.26) in this case are given by (2.18),

$$\mathscr{F}^k_Q(z,\bar{z}) = \sum_{\nu\in\mathbb{Z}} z^{\Delta_{-k-Q/L,\nu L-k}} \Pi_V(z)\, \bar{z}^{\Delta_{-k+Q/L,\nu L-k}} \Pi_V(\bar{z}) \tag{2.55}$$

The character functions (2.19) are now recovered through

$$D^{Lf}_{L^2g}(z,\bar{z}) = \mathscr{F}^{Lf}_{L^2g}(z,\bar{z}) - \mathscr{F}^{Lg}_{L^2f}(z,\bar{z}) = \sum_{s=1}^{\infty} \chi_{L(f+g),Ls}(z)\, \chi_{L(f-g),Ls}(\bar{z}) \tag{2.56}$$

where again f and g are integers with $f>0$ and $-f<g<f$. As in the case of the R-models, the vacuum representation is included in this set for $L=1$ only, and the modular invariant partition function $\mathscr{A}_m(z,\bar{z})$ of (2.32) for $L=1$ (in terms of the sectors defined above) is given by the same expression (2.33) as for the $R=1$ models.

We now switch to *rational* values of $m=u/v$ with coprime positive integers u and v again. Equation (2.54) becomes

$$h = \frac{L^2}{4}\frac{u}{u+v}, \quad \gamma = 1 - \frac{u+v}{L^2u}, \quad l_0 + \nu_0 = \frac{v}{Lu} = \frac{v}{n} \tag{2.57}$$

and the integer n [cf. Eq. (2.35)] is given by $n=Lu$. Once again we define partition functions $\mathscr{G}^k_Q$ in the same way as in Eq. (2.36). If $u(u+v)/n = (u+v)/L$ is an integer (i.e., $u+v$ is a multiple of L), these can be rewritten as follows [cf. Eq. (2.37)]:

$$\mathscr{G}^k_Q(z,\bar{z}) = \sum_{w=0}^{2(u+v)/L-1} \Omega_{-k-Q/L,Lw-k}(z)\, \Omega_{-k+Q/L,Lw-k}(\bar{z}) \tag{2.58}$$

and one obtains expressions bilinear in the characters (2.20) by

$$\begin{aligned} D^{Lf}_{L^2g}(z,\bar{z}) &= D^{L(u-f)}_{L(u-Lg)}(z,\bar{z}) = \mathscr{G}^{Lf}_{L^2g}(z,\bar{z}) - \mathscr{G}^{Lg}_{L^2f}(z,\bar{z}) \\ &= \sum_{w=1}^{(u+v)/L-1} \chi_{L(f+g),Lw}(z)\, \chi_{L(f-g),Lw}(\bar{z}) \end{aligned} \tag{2.59}$$

where f and g are integers satisfying $1 \leqslant Lf \leqslant u-1$ and $|Lg| \leqslant \min\{Lf-1, u-1-Lf\}$.

We now proceed by investigating the additional sectors that occur for the $L=1$ and $L=2$ models (corresponding to the 2_L and 1_L models of ref. 1). Note that since the range of h in (2.12) in the XXZ chain is limited to $h \geqslant 1/4$, the $L=1$ models cannot be realized in the finite-size spectra of the XXZ Heisenberg chain.

2.2.1. The $L=1$ Models. The unitary subset ($v=1$) discussed in ref. 1 corresponds to the tricritical p-state Potts models, $p=[2\cos(\pi/u)]^2$ (2_L-series). Again the structure of the sectors obtained in ref. 1 is identical to what one obtains from Eq. (2.59) for v not restricted to one but integer charge sectors only. Using $n=u$ for $L=1$, the sectors read

$$D_g^f(z,\bar{z})=D_{n-g}^{n-f}(z,\bar{z})=\mathcal{G}_g^f(z,\bar{z})-\mathcal{G}_f^g(z,\bar{z})$$
$$=\sum_{s=1}^{u+v-1}\chi_{f+g,s}(z)\,\chi_{f-g,s}(\bar{z}) \tag{2.60}$$

for any value of $u+v$ and

$$1\leqslant f\leqslant n-1,\qquad |g|\leqslant\min\{f-1,n-1-f\} \tag{2.61}$$

Again we want to include half-integer values for k and Q. Let $\tau=[1-(-1)^{2\tilde{g}}]/4$ as in Eq. (2.41). One obtains as a generalization of Eq. (2.60)

$$D_{\tilde{g}}^{\tilde{f}}(z,\bar{z})=D_{n-\tilde{g}}^{n-\tilde{f}}(z,\bar{z})=\mathcal{G}_{\tilde{g}}^{\tilde{f}}(z,\bar{z})-\mathcal{G}_{\tilde{f}+\tau n}^{\tilde{g}+\tau n}(z,\bar{z})$$
$$=\sum_{s=1}^{u+v-1}\chi_{r,\tilde{f}-\tilde{g}+\tau n}(z)\,\chi_{r,\tilde{f}+\tilde{g}+\tau n}(\bar{z}) \tag{2.62}$$

In this equation, $\tilde{f}$ is integer and $\tilde{g}$ integer or half-integer if u and v are both odd (i.e., for $u+v$ even); for u even and v odd, $\tilde{f}$ and $\tilde{g}$ have to be both even or odd, whereas in the case of an even value of v, $\tilde{g}$ has to be an integer ($\tilde{f}$ possibly being half-integer). We again consider these three cases separately.

a. <u>*$u+v$ Even.*</u> Consider the sectors of Eq. (2.62) with half-integer charge. They are

$$D_{n/2-f}^{-g}(z,\bar{z})=D_{n/2+f}^{n+g}(z,\bar{z})=\mathcal{G}_{n/2-f}^{-g}(z,\bar{z})-\mathcal{G}_{n/2-g}^{-f}(z,\bar{z})$$
$$=\sum_{s=1}^{u+v-1}\chi_{f+g,u+v-s}(z)\,\chi_{f-g,s}(\bar{z}) \tag{2.63}$$

where f and g are integers which comply with Eq. (2.61). Here one always obtains *new* sectors, since there is no possibility to have $u+v=2$, which would be the analog of the case $u=2$ for the $R=1$ models [cf. Eqs. (2.42)–(2.43)].

As a simple example we consider the case $u=3$, $v=1$ with $n=3$, which is a model with central charge $c=1/2$. One has

$$\begin{aligned}D_0^1=D_0^2&=\mathcal{G}_0^1-\mathcal{G}_1^0&&=(0,0)+(\tfrac{1}{16},\tfrac{1}{16})+(\tfrac{1}{2},\tfrac{1}{2})\\ D_{1/2}^0=D_{5/2}^0&=\mathcal{G}_{1/2}^0-\mathcal{G}_{3/2}^2&&=(\tfrac{1}{2},0)+(\tfrac{1}{16},\tfrac{1}{16})+(0,\tfrac{1}{2})\end{aligned} \tag{2.64}$$

and one realizes that taking into account the half-integer charge sectors, one obtains the operator content of the Ising model but with a different distribution of operators into sectors. For instance, the leading thermal exponent is 1/8 for this model, whereas it is 1 for the Ising model which is realized as the corresponding $R=1$ model [see Eq. (2.46) and ref. 1].

b. u Even. The sectors with half-integer values of $\tilde{f}$ and $\tilde{g}$ are given by [cf. Eq. (2.62)]

$$\begin{aligned} D^{-\tilde{g}}_{u/2-\tilde{f}}(z,\bar{z}) = D^{u+\tilde{g}}_{u/2+\tilde{f}}(z,\bar{z}) &= \mathscr{G}^{-\tilde{g}}_{u/2-\tilde{f}}(z,\bar{z}) - \mathscr{G}^{-\tilde{f}}_{u/2-\tilde{g}}(z,\bar{z}) \\ &= \sum_{s=1}^{u+v-1} \chi_{\tilde{f}+\tilde{g},u+v-s}(z)\,\chi_{\tilde{f}-\tilde{g},s}(\bar{z}) \end{aligned} \tag{2.65}$$

which again are *new* sectors compared to Eq. (2.60) provided the half-integer numbers $\tilde{f}$ and $\tilde{g}$ fulfill the same relations as f and g in Eq. (2.61). There is one exception: the models with $u=2$ ($n=2$), which, due to Eq. (2.61), consist of the sector $D^1_0(z,\bar{z})$ alone.

To give an example for this class of models we consider the case $u=2$, $v=3$ (hence $n=2$), which corresponds to a central charge $c=-22/5$. The only sector is

$$D^1_0 = \mathscr{G}^1_0 - \mathscr{G}^0_1 = 2\cdot(0,0) + 2\cdot(-\tfrac{1}{5},-\tfrac{1}{5}) \tag{2.66}$$

As an example for a model with new sectors we choose $u=4$, $v=1$ ($n=4$, $c=7/10$), which corresponds to the tricritical Ising model.[1] We obtain

$$\begin{aligned} D^1_0 = D^3_0 &= \mathscr{G}^1_0 - \mathscr{G}^0_1 &&= (0,0) + (\tfrac{1}{10},\tfrac{1}{10}) + (\tfrac{3}{5},\tfrac{3}{5}) + (\tfrac{3}{2},\tfrac{3}{2}) \\ D^2_0 &= \mathscr{G}^2_0 - \mathscr{G}^0_2 &&= 2\cdot(\tfrac{3}{80},\tfrac{3}{80}) + 2\cdot(\tfrac{7}{16},\tfrac{7}{16}) \\ D^2_1 = D^2_3 &= \mathscr{G}^2_1 - \mathscr{G}^1_2 &&= (\tfrac{3}{2},0) + (\tfrac{3}{5},\tfrac{1}{10}) + (\tfrac{1}{10},\tfrac{3}{5}) + (0,\tfrac{3}{2}) \\ D^{1/2}_{1/2} = D^{7/2}_{7/2} &= \mathscr{G}^{1/2}_{1/2} - \mathscr{G}^{3/2}_{3/2} &&= (0,\tfrac{7}{16}) + (\tfrac{1}{10},\tfrac{3}{80}) + (\tfrac{3}{5},\tfrac{3}{80}) + (\tfrac{3}{2},\tfrac{7}{16}) \\ D^{7/2}_{1/2} = D^{1/2}_{7/2} &= \mathscr{G}^{7/2}_{1/2} - \mathscr{G}^{5/2}_{3/2} &&= (\tfrac{7}{16},0) + (\tfrac{3}{80},\tfrac{1}{10}) + (\tfrac{3}{80},\tfrac{3}{5}) + (\tfrac{7}{16},\tfrac{3}{2}) \end{aligned} \tag{2.67}$$

which defines a model which differs from the corresponding $R=1$ model [cf. Eq. (2.44)].

c. v Even. At last we again turn to the case of even values for v. As for the $R=1$ models, there are no new sectors for half-integer values of $\tilde{f}$ in Eq. (2.62) because these do not result in new boundary conditions, due to the identity $(k+u/2)\,l_0 = kl_0 + v/2 \equiv kl_0 \bmod 1$.

2.2.2. The $L=2$ Models. As above, the discussion of the $L=2$ models is analogous to that for the $R=2$ models. The unitary subset corresponds to the $O(p)$ models, $p=2\cos(\pi/u)$. Only the integer charge sectors for u odd $(v=1)$ were discussed in ref. 1 (1_L-models). If $u+v$ is even, Eq. (2.58) applies and from Eq. (2.59) one finds

$$D_{4g}^{2f}(z,\bar{z})=D_{2(u-2g)}^{2(u-f)}(z,\bar{z})=\mathscr{G}_{4g}^{2f}(z,\bar{z})-\mathscr{G}_{4f}^{2g}(z,\bar{z})$$
$$=\sum_{w=1}^{(u+v)/2-1}\chi_{2(f+g),2w}(z)\,\chi_{2(f-g),2w}(\bar{z}) \tag{2.68}$$

Additional sectors are obtained as follows.

a. *(u+v) Even.* One obtains

$$\mathscr{G}_Q^k(z,\bar{z})=\sum_{d=1-\xi}^{(u+v)/2-\xi}\Omega_{k+Q/2+\eta u,\,\xi+2d}(z)\,\Omega_{k-Q/2+\eta u,\xi+2d}(\bar{z})$$
$$+\sum_{d=0}^{(u+v)/2-1}\Omega_{k+Q/2+\eta u,\,-(\xi+2d)}(z)\,\Omega_{k-Q/2+\eta u,\,-(\xi+2d)}(\bar{z}) \tag{2.69}$$

where $\eta=[1-(-1)^Q]/4$ and $\xi=[1-(-1)^k]/2$ $\{\xi=[1-(-1)^{k+Q}]/2\}$ for $u+v\equiv 0 \bmod 4$ $(u+v\equiv 2 \bmod 4)$, respectively. The following additional sectors appear:

$$D_{4g+t}^{2f+\omega}(z,\bar{z})=\mathscr{G}_{4g+t}^{2f+\omega}(z,\bar{z})-\mathscr{G}_{2(2f+\omega)+tu}^{(4g+t)/2+\varepsilon((2\omega+t)/2)u}(z,\bar{z}) \tag{2.70}$$

for $\omega\in\{0,1\}$, $t\in\{0,1,2,3\}$, and $\varepsilon=(-1)^{(u+1)/2}$. They are given as a bilinear expression in Virasoro characters as follows:

$$D_Q^k(z,\bar{z})=\sum_{d=1-\xi}^{(u+v)/2-1}\chi_{k+Q/2+\eta u,\xi+2d}(z)\,\chi_{k-Q/2+\eta u,\xi+2d}(\bar{z}) \tag{2.71}$$

Of course, the character functions that enter in Eq. (2.71) have to comply with the conditions given by Eq. (2.22) in order to obtain physical sectors.

b. *(u+v) Odd.* Here one again has to combine two sectors $\mathscr{G}_Q^k(z,\bar{z})$ as in the $R=2$ case [cf. Eq. (2.52)]. The partition functions $\tilde{\mathscr{G}}_Q^k(z,\bar{z})=\mathscr{G}_Q^k(z,\bar{z})+\mathscr{G}_Q^{k+u}(z,\bar{z})$ coincide with the sectors $\mathscr{G}_{Q/2}^k(z,\bar{z})$ [see Eq. (2.58)] of the corresponding $L=1$ model with $n=u$.

This completes our discussion of the projection mechanism in the finite-size scaling limit. We now turn our attention to chains of *finite* length N and to their spectra.

3. PROJECTION MECHANISM FOR FINITE SYSTEMS

We commence this section by explaining the sense in which the projection mechanism which we so far have only established for the continuum limit can also be applied to finite systems. For this purpose we try to give a meaning to differences of partition functions for a finite number of sites N.

Consider a general sector

$$D^k_Q(z, \bar{z}) = \mathscr{G}^k_Q(z, \bar{z}) - \mathscr{G}^{k'}_{Q'}(z, \bar{z}) \tag{3.1}$$

The finite-size analog of this equation is

$$D^k_Q(z, \bar{z}, N) = \mathscr{G}^k_Q(z, \bar{z}, N) - \mathscr{G}^{k'}_{Q'}(z, \bar{z}, N) \tag{3.2}$$

where the $\mathscr{G}^k_Q(z, \bar{z}, N)$ are defined[5] by summing up the finite-size partition functions $\mathscr{F}^k_{Q+n\mu}(z, \bar{z}, N)$ of (2.26) with the appropriate value of n. We want to interpret the function $D^k_Q(z, \bar{z}, N)$ defined by Eq. (3.2) as the partition function of a projected system. This can be done provided that *any* level which contributes to the partition function $\mathscr{G}^{k'}_{Q'}(z, \bar{z}, N)$ has a correspondent in $\mathscr{G}^k_Q(z, \bar{z}, N)$, i.e., for any eigenstate of the *XXZ* Heisenberg chain contributing to $\mathscr{G}^{k'}_{Q'}(z, \bar{z}, N)$ there is at least one eigenstate with the same energy and momentum which contributes to $\mathscr{G}^k_Q(z, \bar{z}, N)$. In this case the difference $D^k_Q(z, \bar{z}, N)$ of the two partition functions $\mathscr{G}^k_Q(z, \bar{z}, N)$ and $\mathscr{G}^{k'}_{Q'}(z, \bar{z}, N)$ is the partition function of a system consisting only of those states which are left over if one eliminates all the degenerate doublets with one state in each sector. Denoting, in analogy to the notation of Appendix A [cf. Eqs. (A.8) and (A.14)], the set of all pairs of energy and momentum eigenvalues[6] that contribute to $\mathscr{G}^k_Q(z, \bar{z}, N)$ by $\mathscr{G}^k_Q(N)$, the condition means that the sets $\mathscr{G}^k_Q(N)$ and $\mathscr{G}^{k'}_{Q'}(N)$ should satisfy the inclusion

$$\mathscr{G}^k_Q(N) \supset \mathscr{G}^{k'}_{Q'}(N) \tag{3.3}$$

If this is true, the projection mechanism therefore actually *defines* a finite-size model (in the sense that it determines the spectrum) as long as we are in the physical region $h \geqslant 1/4$ [see (2.12)] of the *XXZ* Heisenberg chain.

[5] This is true for minimal models; of course the generic case would be obtained by replacing $\mathscr{G}^k_Q$ by $\mathscr{E}^k_Q$ throughout this section.

[6] There is a small difference from the definitions used in Appendix A since we subtracted a suitably chosen ground-state energy throughout Section 2. For obvious reasons, however, this does not affect the arguments for degeneracies of eigenvalues of finite chains. But note that the index k denotes different boundary conditions depending on the type of model considered, whereas in Appendix A the boundary condition for $\hat{\mathscr{G}}^K_Q(N)$ is given by $\alpha = q^{2K}$ [cf. (A.5)].

This condition is fulfilled for all R-models and for the $L>1$ models where $h \geqslant 1/4$ if $m \geqslant 1/(L^2-1)$, but it for instance excludes all $L=1$ models.

In Appendix A we establish intertwining relations between charge sectors of the XXZ Hamiltonian with different toroidal boundary conditions using powers of the quantum algebra generators of $U_q[sl(2)]$ [the corresponding representation on $\mathscr{H}(N)$ is given by Eq. (A.2)]. Analogous intertwining relations hold for the corresponding translation operators (2.4). These relations allow us to obtain inclusion relations for the sets of simultanous eigenvalues of the Hamiltonian and the translation operator of the form (3.3). The main results obtained in Appendix A are Eq. (A.10) for generic values of q and Eq. (A.13) if q is a root of unity. We proceed by having a closer look at the R- and L-models separately.

3.1. The *R*-Models

For the R-models, the connection between the anisotropy γ and the boundary condition is determined by the equations [cf. Eqs. (2.28) and (2.29)]

$$\begin{aligned} q &= -\exp(-i\pi\gamma) = \exp\left(i\pi \frac{m}{R^2(m+1)}\right) \\ \alpha(k) &= \exp[2\pi i k(l_0+\nu_0)] = \exp\left(\frac{2\pi i k}{R(m+1)}\right) \end{aligned} \tag{3.4}$$

where m and k are arbitrary. It follows that

$$\alpha(k) = e^{2\pi i k/R} q^{-2Rk} \tag{3.5}$$

The phase factor in Eq. (3.5) is equal to one if k is an integer multiple of R. In this case one obtains with $k=Rf$, f integer, the relation

$$\alpha(Rf) = q^{-2R^2 f} \tag{3.6}$$

i.e., if $k=Rf$, then the corresponding value of K in the notation of Appendix A would be $K=-R^2 f=-Rk$. Comparing now the inclusion relations given in Eqs. (A.10) and (A.13) (choosing $K=-R^2 f$ and $Q=R^2 g$ with integers f and g fulfilling $0 \leqslant |g| \leqslant |f|$) with the sectors (2.31) [respectively (2.38)], one immediately realizes that that for all R-models and all the sectors considered the inclusion relation (3.3) is fulfilled and thus the projection mechanism extends to finite chains.

We now focus on the minimal case, i.e., $m=u/v$ with positive coprime integers u and v. In Section 2 we explicitly obtained additional sectors for the $R=1$ (Section 2.1.1) and the $R=2$ (Section 2.1.2) models. In ref. 1 it was observed numerically that in the case of the unitary minimal series (i.e., $v=1$) for the $R=2$ models[7] only part of the sectors given in Eqs. (2.50)–(2.51) show the degeneracies (3.3). In particular, this is true for all sectors $D^k_Q(z,\bar{z})$ of Eq. (2.51) with $\xi=0$, which are the sectors $D^k_Q(z,\bar{z})$ with even k (even $k+Q$) for $u\equiv 0 \bmod 4$ ($u\equiv 2 \bmod 4$), respectively.

To understand these observations, let us consider values of u which are multiples of R, i.e., $u\equiv 0 \bmod R$. This gives $q^{\pm R(u+v)}=(-1)^{u/R}$ and hence Eq. (3.5) can be modified as follows:

$$\alpha(k)=(-1)^{u/R}\, e^{2\pi i k/R} q^{-2(Rk\pm R(u+v)/2)} \tag{3.7}$$

This means that if u/R is even, i.e., $u\equiv 0 \bmod 2R$, one obtains as a generalization of Eq. (3.6) for $k=R\tilde{f}$, $\tilde{f}$ integer, the equation

$$\alpha(R\tilde{f})=q^{-2R^2\tilde{f}}=q^{-2(R^2\tilde{f}\pm R(u+v)/2)} \tag{3.8}$$

Analogously, if u/R is odd, i.e., $u\equiv R \bmod 2R$, one obtains with $k=R\tilde{f}$ in addition to Eq. (3.6), but now for *half-integer* values of $\tilde{f}$ (i.e., $2\tilde{f}$ is an *odd* integer), the relation

$$\alpha(R\tilde{f})=-q^{-2R^2\tilde{f}}=q^{-2(R^2\tilde{f}\pm R(u+v)/2)} \tag{3.9}$$

The inclusion relation (A.13) with $K=-R^2\tilde{f}\pm R(u+v)/2$ and $Q=R^2\tilde{g}\pm R(u+v)/2$ now results in the relation

$$\mathcal{G}^{R\tilde{f}}_{R^2\tilde{g}\pm R(u+v)/2}(N)\supset \mathcal{G}^{R\tilde{g}}_{R^2\tilde{f}\pm R(u+v)/2}(N) \tag{3.10}$$

with appropriate values of $\tilde{f}$ and $\tilde{g}$. From this one deduces that in fact *all* the additional sectors [see Eqs. (2.42) and (2.45)] for the $R=1$ models show the degeneracies (3.3), whereas for the $R=2$ models this is true for all sectors $D^k_Q(z,\bar{z})$ of (2.51) with even values of k if $u\equiv 0 \bmod 4$ [respectively for all sectors $D^k_Q(z,\bar{z})$ of (2.51) with even $k+Q$ if $u\equiv 2 \bmod 4$] (for arbitrary value of v).

3.2. The *L*-Models

The discussion of the L-models is completely analogous to that of the previous section on the R-models. One should keep in mind that results on the finite-chain spectra only apply if $h\geqslant 1/4$ [see (2.12)]. The connection between the anisotropy γ and the boundary condition is given by (2.54)

[7] Remember that these are the 1_R models in the notation of ref. 1.

$$q = -\exp(-i\pi\gamma) = \exp\left(i\pi \frac{m+1}{L^2 m}\right)$$
$$\alpha(k) = \exp[2\pi i k(l_0 + v_0)] = \exp\left(\frac{2\pi i k}{Lm}\right) \tag{3.11}$$

and hence

$$\alpha(k) = e^{-2\pi i k/L} q^{2Lk} \tag{3.12}$$

Considering values of k which are integer multiples of L, i.e., $k = Lf$, f integer, one obtains

$$\alpha(Lf) = q^{2L^2 f} \tag{3.13}$$

i.e., the corresponding value of K in the notation of Appendix A would be $K = L^2 f = Lk$. Again, the inclusion relations (A.10) and (A.13), now with the choices $K = L^2 f$ and $Q = L^2 g$ (f, g integers fulfilling $0 \leqslant |g| \leqslant |f|$), guarantee that all for all sectors given by Eqs. (2.56) [respectively (2.59)] the projection mechanism can be applied for finite-size systems.

Finally, we again consider the minimal case $m = u/v$, where u and v are coprime positive integers. Here we consider models where $u + v \equiv 0 \bmod L$. Using $q^{\pm Lu} = (-1)^{(u+v)/L}$, one can modify Eq. (3.12) as follows:

$$\alpha(k) = (-1)^{(u+v)/L} e^{-2\pi i k/L} q^{2(Lk \pm Lu/2)} \tag{3.14}$$

If $(u+v)/L$ is even, i.e., $u + v \equiv 0 \bmod 2L$, one obtains for $k = L\tilde{f}$, $\tilde{f}$ integer, the expression

$$\alpha(L\tilde{f}) = q^{2L^2\tilde{f}} = q^{2(L^2\tilde{f} \pm Lu/2)} \tag{3.15}$$

whereas if $(u+v)/L$ is odd, i.e., $u + v \equiv L \bmod 2L$, substituting $k = L\tilde{f}$ with *half-integer* values of $\tilde{f}$ into Eq. (3.14) yields

$$\alpha(L\tilde{f}) = -q^{2L^2\tilde{f}} = q^{2(L^2\tilde{f} \pm Lu/2)} \tag{3.16}$$

We now use the inclusion relation (A.13) with $K = L^2\tilde{f} \pm Lu/2$ and $Q = L^2\tilde{g} \pm Lu/2$ and obtain the relation

$$\mathscr{G}^{L\tilde{f}}_{L^2\tilde{g} \pm Lu/2}(N) \supset \mathscr{G}^{L\tilde{g}}_{L^2\tilde{f} \pm Lu/2}(N) \tag{3.17}$$

where again $\tilde{f}$ and $\tilde{g}$ have to be chosen appropriately. For the $L = 2$ models (note that the $L = 1$ models all have $h < 1/4$), one finds the following behavior. The sectors $D^k_Q(z, \bar{z})$ in Eq. (2.71) possess the finite-size degeneracies (3.3) if $\xi = 0$, which means if k $(k + Q)$ is even for $u + v \equiv 0$

mod 4 ($u+v \equiv 2 \bmod 4$). This proves and extends the numerical results of ref. 1.

Let us summarize the results of this section. We explicitly derived all the degeneracies observed numerically in ref. 1 using the results of Appendix A. In fact, there is a lot more information in our equations. In particular, we could show that for the $R=1$ models all sectors (including those with half-integer charges) show the required degeneracies (3.3) to define a finite-size model. In ref. 1 it has been shown (by numerical comparison) that the spectrum of the $R=1$ model with $u=3$, $v=1$ (central charge $c=1/2$) is exactly that of the Ising quantum chain and that the $R=1$ model with $u=5$, $v=1$ ($c=4/5$) reproduces the spectrum of the 3-state Potts quantum chain (see ref. 21 and references therein), in both cases with toroidal boundary conditions. In the next section we address the question of what the new sectors in these $R=1$ models correspond to in the Ising and 3-state Potts quantum chains.

4. INTERPRETATION OF THE NEW SECTORS

4.1. New Boundary Conditions for the Ising and 3-State Potts Quantum Chains

In this section we discuss the significance of the new half-integer charge sectors in the projected systems and show that they are related to a new kind of boundary condition in these models. Before we do so we want to make the problem at hand more precise by reminding the reader of the relation between the labels k and q in the partition functions D_q^k and the projected systems to which they correspond.

As an example, consider the two-dimensional Ising model on a torus. In the extreme anisotropic limit it is described by the lattice Hamiltonian[22]

$$H = -\tfrac{1}{2} \sum_{j=1}^{M} (\sigma_j^x + \lambda \sigma_j^z \sigma_{j+1}^z) \tag{4.1}$$

Here, M represents the number of sites and λ plays the role of the inverse temperature. In the thermodynamic limit $M \to \infty$ the model has a critical point at $\lambda = 1$, where it is described by a conformal field theory with central charge $c=1/2$ (a Majorana fermion). The Ising model has a global $\mathbb{Z}_2$ symmetry (the spin-flip operation); as a consequence H commutes with the operator

$$S = \prod_{j=1}^{M} \sigma_j^x \tag{4.2}$$

with eigenvalues $S = \pm 1$ splitting H into two sectors, one of which is even under this operation ($S=1$), the other one odd ($S=-1$). On the other hand, it is known that for quantum chains whose group of global symmetries is of order n, there are also n different types of toroidal boundary conditions, i.e., boundary conditions compatible with the geometry of a torus.[23] Here $n=2$ and one has periodic boundary conditions ($\sigma^z_{M+1} = \sigma^z_1$) and antiperiodic boundary conditions ($\sigma^z_{M+1} = -\sigma^z_1$). In the two-dimensional model from which H is obtained the latter correspond to a seam of antiferromagnetic bonds in an otherwise ferromagnetic system. The choice of boundary conditions is reflected in the structure of the translation operator in a very intuitive way: With periodic boundary conditions, (4.1) commutes with the translation operator

$$T = \prod_{j=1}^{M-1} P_j \tag{4.3}$$

with P_j defined in (2.4). On the other hand, in the case of antiperiodic boundary conditions an additional spin-flip operation at the boundary is necessary in order to construct a commuting translation operator T':

$$T' = T\sigma^x_M \tag{4.4}$$

One finds $T^M = 1$ and $T'^M = S$.

To conclude this short reminder of the Ising quantum chain, let us denote the eigenvalues of (4.1) by $\varepsilon^{l'}_{l;j}$, $l=0$ ($l=1$) in the even (odd) sector, $l'=0$ ($l'=1$) for periodic (antiperiodic) boundary conditions, and $j = 1,..., 2^{M-1}$ according to the number of states in these sectors.

As already noted, the numbers k and q in the partition functions D^k_q of the projected systems label the sectors of these models according to their internal global symmetries and the type of toroidal boundary condition imposed on the systems. The discussion of the $R=1$ model with $u=3$ and $v=1$ in Section 2 in the thermodynamic limit $M \to \infty$ amounts to the statement that at the critical point $\lambda = 1$ the projected sectors (2.46) of the XXZ Heisenberg chain coincide with the sectors described here (in the limit $N \to \infty$). In particular, the levels contributing to D^1_0 are the scaled energy gaps $\bar{\varepsilon}^0_{0;j}$ [here $\bar{\varepsilon}^{l'}_{l;j} = M/2\pi(\varepsilon^{l'}_{l;j} - \varepsilon^0_{0;1})$], to D^2_0 contribute $\bar{\varepsilon}^0_{1;j}$ and $\bar{\varepsilon}^1_{0;j}$, which are degenerate, and to D^2_1 contribute $\bar{\varepsilon}^1_{1;j}$.

This raises the question: To which boundary conditions and sectors of the Ising model do the sectors $D^{1/2}_{1/2}$ and $D^{7/2}_{1/2}$ of Eq. (2.46) correspond? The existence of these sectors is actually surprising, as the spin-flip symmetry allows only for the periodic and antiperiodic toroidal boundary conditions for H discussed above and therefore no new toroidal boundary conditions should be expected. This observation allows us to formulate the problem: One has to find an additional symmetry of the Ising model and the corresponding boundary conditions which give the spectrum corresponding the sectors $D^{1/2}_{1/2}$ and $D^{7/2}_{1/2}$ of (2.46).[8]

In order to answer this question, we note that even for *finite* systems the scaled energy gaps $\bar{\varepsilon}^{l'}_{l;j}$ of the Ising Hamiltonian with M sites are identical to those scaled energy gaps of the XXZ Heisenberg chain with $N=2M$ sites which contribute to the integer charge sectors D^k_q of (2.46). The reason for this important observation is studied in refs. 24–26 and we do not repeat the discussion here. In ref. 26 we construct lattice Hamiltonians for the Ising and 3-state Potts models which have the spectrum contributing to the new partition functions of the XXZ Heisenberg chain with $N=2M-1$ sites (see below). Given these Hamiltonians, we can analyze their symmetries and the physical significance of the boundary conditions involved.

4.1.1. Ising Model. A Hamiltonian such that its scaled energy gaps coincide with those obtained through the projection mechanism from the normalized XXZ Heisenberg chain (2.11) with $q=-\exp(i\pi/4)$, $\alpha=-q^{-2k}$, and $N=2M-1$ sites[9] is given by[(26)]

$$\tilde{H}=-\sum_{j=1}^{2M-1}(e_j-\tfrac{1}{2}) \tag{4.5}$$

with

$$e_{2j-1}=\tfrac{1}{2}(1+\sigma^x_j), \qquad e_{2j}=\tfrac{1}{2}(1+\sigma^z_j\sigma^z_{j+1}) \tag{4.6}$$

$$e^{\pm}_{2M-1}=\tfrac{1}{2}(1\pm\sigma^y_M\sigma^z_1) \tag{4.7}$$

where $1\leqslant j\leqslant M-1$. Here $\tilde{H}$ is an Ising Hamiltonian acting on a chain of M sites with a boundary term $\pm\sigma^y_M\sigma^z_1$. Note that, as opposed to periodic and antiperiodic boundary conditions as well as to the "generalized defects" investigated in ref. 27 (which include boundary couplings of the form $\sigma^y_M\sigma^z_1$), there is no operator σ^x_M present. To the best of our knowledge this type of boundary condition has not yet been studied in the literature. The spectrum of $\tilde{H}$ leads to the partition functions (2.46) for half-integer charge sectors. The highest weight representations of the Virasoro algebra (0, 1/16), (1/16, 0), (1/2, 1/16), and (1/16, 1/2) contributing to these partition functions represent the anomalous dimensions of some new spinor fields.

The two different signs in the boundary term amount to complex

[8] The Ising Hamiltonian (4.1) is also invariant under a parity operation, but this is of no importance to the present discussion.

[9] The scaling factor $N/2\pi$ in (2.15) has to be changed into M/π if $N=2M-1$.

conjugation. Since the Hamiltonian (4.5) is Hermitian, the corresponding spectra are identical. This is indeed observed in the projected spectra: The eigenvalues of the *XXZ* Heisenberg chain contributing to the sectors $D^{1/2}_{1/2}$ and $D^{7/2}_{1/2}$, respectively, are related through complex conjugation of the boundary angle α in (2.1). Since for the choice of q and α under consideration the *XXZ* Heisenberg chain is Hermitian and all eigenvalues are therefore real, the two spectra are degenerate.

In order to understand the symmetry from which the new boundary conditions arise, we go back to the standard Ising model defined by the Hamiltonian (4.1). First we want to stress again that the eigenvalues of H in (4.1) with M sites and $\lambda=1$ are obtained through the projection mechanism from the *XXZ* Heisenberg chain (2.1) with $N=2M$ sites. The eigenstates of the *XXZ* Heisenberg chain are also eigenstates of the translation operator (2.4), which has $2M$ different eigenvalues of the form $\exp[(2\pi ik+i\pi\phi)/2M]$, $0\leqslant k\leqslant 2M-1$. As shown in Appendix A, the projected eigenstates of (2.1) remain eigenstates also of (2.4). On the other hand, the translation operators (4.3) and (4.4) of the Ising model have only M different eigenvalues of the form $\exp[(2\pi ik+i\pi l)/M]$, $0\leqslant k\leqslant M-1$ (with $l=0, 1$, depending on the sector). From this observation we learn that T and T' of the Ising model (4.1) are *not* the equivalent operators to the translation operator $T(\alpha, 2M)$ of the *XXZ* Heisenberg chain, but that they correspond to $T^2(\alpha, 2M)$. Technically speaking this means that on the projected subspace of the *XXZ* Heisenberg chain under consideration the translation operators T and T' are not representations of the translation operator $T(\alpha, 2M)$, but os its square $T^2(\alpha, 2M)$[10]. But since the projected eigenstates are also eigenstates of $T(\alpha, 2M)$, we have established the presence of an additional symmetry in the Ising model besides the $\mathbb{Z}_2$ spin-flip symmetry and translational invariance.

The physical meaning of the symmetry generated by $T(\alpha, 2M)$ in the projected system can be understood by studying its representation D in the Ising model. One finds that D satisfies[25,26]

$$De_{j'}=e_{j'+1}D \tag{4.8}$$

where the $e_{j'}$, $1\leqslant j'\leqslant 2M$, are defined as in Eq. (4.6) by extending the range of j to $1\leqslant j\leqslant M$. This is the duality transformation D (see Appendix B). We conclude that the additional symmetry we found is duality, which, at (and only at) the self-dual point $\lambda=1$, is indeed a true symmetry: If $\lambda=1$, the duality relation (B.3) becomes

$$H^D(1)=DH(1)\,D^{-1}=H(1) \tag{4.9}$$

[10] This explains why we had to neglect macroscopic momenta π in the spectrum of the *XXZ* Heisenberg chain to get the momenta in the Ising chain

Since D^2 corresponds to translations (see Appendix B), we can say that duality is the "square root" of translations. Obviously, not only the standard Ising Hamiltonian (4.1), but also $\tilde{H}$ (4.5) commutes with the duality transformation (in an appropriate representation $\tilde{D}$).[11]

Having found an additional symmetry in the Ising model at its self-dual point and the Hamiltonian (4.5) giving rise to the new sectors discovered through the projection mechanism, we can look for an explicit representation of the translation operator commuting with (4.5). As in the standard case of periodic and antiperiodic boundary conditions, this sheds light on the physical meaning of the boundary conditions. One finds, corresponding to the two possible choices of the sign in e_{2M-1}, the translation operator $\tilde{T}$ and its complex conjugate $\tilde{T}^*$ commuting with $\tilde{H}$ and $\tilde{H}^*$, respectively,

$$\tilde{T} = T g_{2M-2}\, g_{2M-1} \tag{4.10}$$

with T given in (4.3). The operators g_j are related to the duality transformation (B.4) and defined by

$$g_{2M-2} = -\frac{1-i}{2}(1 - i\sigma^z_{M-1}\sigma^z_M), \quad g_{2M-1} = -\frac{1-i}{2}(1 - i\sigma^x_M) \tag{4.11}$$

A straightforward calculation shows that the Mth power of $\tilde{T}$ is the duality transformation in each sector (see Appendix B). This clarifies the meaning of this type of boundary condition: $\tilde{T}$ performs the local equivalent of the duality transformation at the boundary in addition to a pure translation. The Mth power of $\tilde{T}$ gives the symmetry operator to which the boundary condition is related. This exhibits its relation to the duality symmetry in the same way as the existence of the spin-flip symmetry resulted in the existence of the translation operator T' acting locally at the boundary as spin-flip operator times a global translation [cf. also the translation operator (2.4) for the XXZ chain with toroidal boundary conditions].

4.1.2. 3-State Potts Model. The 3-state Potts model is obtained from the $R=1$ series with $u=5$ and $v=1$. The discussion of the new boundary conditions here is in complete analogy to the previous discussion of the Ising model and we state only the results.

Taking the extreme anisotropic limit of the transfer matrix of the

[11] Strictly speaking, this statement applies to the mixed sector versions of (4.1) and (4.5) discussed in Appendix B. Here we have omitted all subscripts and superscripts specifying the various sectors and boundary conditions in the Ising model and hence the representation of D. A detailed discussion is given in Appendix B.

3-state Potts model on a torus with periodic boundary conditions, one obtains the Hamiltonian[21]

$$H = -\frac{2}{3\sqrt{3}} \sum_{j=1}^{M} \Gamma_j + \Gamma_j^2 + \lambda(\sigma_j \sigma_{j+1}^2 + \sigma_j^2 \sigma_{j+1}) \tag{4.12}$$

Here Γ_j and σ_j are the matrices

$$\Gamma_j = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_j, \qquad \sigma_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}_j \tag{4.13}$$

acting on site j and $\omega = \exp(2\pi i/3)$. Again λ plays the role of the inverse temperature and in the thermodynamic limit $M \to \infty$ the model has a critical point at $\lambda = 1$ with central charge $c = 4/5$. The Hamiltonian (4.12) with periodic boundary conditions $\sigma_{M+1} = \sigma_1$ is symmetric under the permutation group S_3 and commutes with the operators Z and E defined by

$$Z = \prod_{j=1}^{M} \Gamma_j, \qquad E = \prod_{j=1}^{M} V_j, \qquad V_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_j \tag{4.14}$$

They satisfy $Z^3 = E^2 = 1$ and $EZ = Z^2 E$. The Hamiltonian H splits into four sectors, $H_{0,+}$, $H_{0,-}$, H_1, and H_2, corresponding to the four irreducible representations of S_3. They are labeled according to the eigenvalues ω^κ, $\kappa = 0, 1, 2$, of Z and ± 1 of E, respectively.

According to the S_3 symmetry there are nonperiodic toroidal boundary conditions[23] accounting for the various integer charge sectors (2.47) obtained from the XXZ Heisenberg chain with $N = 2M$ sites. At the self-dual point $\lambda = 1$ the symmetry is enhanced as the duality transformation becomes a true symmetry of the (mixed sector) model (see Appendix B).

A Hamiltonian for the 3-state Potts model such that its scaled energy gaps coincide with those obtained through the projection mechanism from the normalized XXZ Heisenberg chain (2.11) with $q = -\exp(i\pi/6)$, $\alpha = -q^{-2k}$, and $N = 2M - 1$ sites is given by[26]

$$\tilde{H} = -\frac{2}{\sqrt{3}} \sum_{j=1}^{2M-1} \left(e_j - \frac{1}{3} \right) \tag{4.15}$$

Here the operators e_j are defined by

$$\begin{aligned} e_{2j-1} &= \tfrac{1}{3}(1 + \Gamma_j + \Gamma_j^2), \qquad e_{2j} = \tfrac{1}{3}(1 + \sigma_j \sigma_{j+1}^2 + \sigma_j^2 \sigma_{j+1}) \\ e_{2M-1}^{(\kappa)} &= \tfrac{1}{3}(1 + \omega^{-k} \Gamma_M^{\kappa} \sigma_M \sigma_1^2 + \omega^{\kappa} \sigma_M^2 \Gamma_M^{-\kappa} \sigma_1) \end{aligned} \tag{4.16}$$

where $1 \leqslant j \leqslant M-1$ and $\kappa = 1, 2$. Note that $e^{(1)}_{2M-1}$ is the complex conjugate of $e^{(2)}_{2M-1}$. Since the Hamiltonian $\tilde{H}$ of (4.15) is Hermitian, the spectra for $\kappa = 1$ and $\kappa = 2$ are identical. This corresponds to the degeneracy of the energy levels contributing to the partition functions $D^{1/2}_{1/2}$ and $D^{11/2}_{1/2}$, $D^{3/2}_{1/2}$ and $D^{9/2}_{1/2}$, and $D^{1/2}_{3/2}$ and $D^{11/2}_{3/2}$ [see Eq. (2.47)]. These pairs of partition functions are obtained through complex conjugation of the boundary angle α of the *XXZ* Heisenberg chain, which is also Hermitian. Furthermore, analysis of the number of eigenstates in each sector of $\tilde{H}$ for finite values of M[7,26] shows that $\tilde{H}_{0,+}$ contains the energy levels contributing to $D^{3/2}_{1/2}$, that $\tilde{H}_{0,-}$ contains the energy levels contributing to $D^{1/2}_{3/2}$, and that $\tilde{H}_1 \cong \tilde{H}_2$ contains the energy levels contributing to $D^{1/2}_{1/2}$. In the 3-state Potts model the new boundary conditions are related to the duality transformation in the same way as in the Ising model.

4.2. Numerical Results for the $R=1$ Model with $u=2$ and $v=3$

As already mentioned in Section 2, the minimal $R=1$ models with $u=2$ have a special feature: one obtains the same character functions by the projection procedure described in Section 2 applied to the *XXZ* chain with an even or with an odd number of sites [see Eqs. (2.39) and (2.42)].

Table I. Ground-State Energy per Site of the $R=1$ Model with $u=2$, $v=3$ and Chains with up to 18 Sites

N	$-E_0(N)/N$	$-E_0(N)/N - A_0 - \pi c/6N^2$
2	−0.292428	−0.081784
3	0.097476	−0.011858
4	0.217963	−0.003362
5	0.271846	−0.001316
6	0.300700	−0.000620
7	0.317968	−0.000330
8	0.329126	−0.000192
9	0.336753	−0.000119
10	0.342199	−0.000078
11	0.346222	−0.000053
12	0.349279	−0.000037
13	0.351656	−0.000027
14	0.353541	−0.000020
15	0.355061	−0.000015
16	0.356304	−0.000012
17	0.357334	−0.000009
18	0.358197	−0.000007

This behavior is obviously different from the one observed in the cases of the Ising model ($u=3, v=1$) and the 3-state Potts model ($u=5, v=1$) above. It is our conjecture that for these models it is not necessary to distinguish between spectra obtained from an even or an odd number of sites. The same phenomenon has been observed for the projection mechanism for open boundary conditions.[(6)] At this place we want to present some numerical results for the simplest example of this series, namely the model with $v=3$ ($n=5$) with central charge $c=-22/5$. The sectors for this model are given in Eq. (2.43).

First we discuss the ground-state energies. As ground state of our projected model we choose $E_0(N)=E^1_{0;2}(3/5, N)$ for an even number of sites N and $E_0(N)=E^0_{3/2;1}(3/5, N)$ for an odd number of sites N. These are the states which in the finite-size scaling limit give a contribution of one to the partition functions $\mathscr{G}^1_0(z,\bar z)$ and $\mathscr{G}^0_{3/2}(z,\bar z)$ for even and odd number of sites, respectively (see Eq. (2.43)). In Table I we list the numerical values of $-E_0(N)/N$ for up to 18 sites. Also shown are the differences between these

Table II. The Lowest Scaled Energy Gaps That Contribute to the Sectors $D^1_0(z,\bar z)$, resp. $D^0_{3/2}(z,\bar z)$, and D^2_1, resp. $D^4_{1/2}$, for Chains of Length N[a]

N	$\chi_{1,1}(z)\,\chi_{1,1}(\bar z)$		$\chi_{1,1}(z)\,\chi_{1,2}(\bar z)$			
2			−0.248 220			
3			−0.216 948	0.775 444		
4	1.606 516		−0.208 906	0.803 258	1.318 981	
5	1.816 384		−0.205 547	0.808 298	1.554 938	1.565 539
6	1.909 047	2.281 378	−0.203 801	0.808 328	1.664 745	1.678 934
7	1.952 922	2.552 745	−0.202 771	0.807 313	1.719 916	1.734 239
8	1.975 088	2.712 909	−0.202 111	0.806 210	1.749 707	1.762 924
9	1.986 887	2.810 122	−0.201 662	0.805 245	1.766 826	1.778 613
10	1.993 425	2.870 984	−0.201 343	0.804 447	1.777 197	1.787 578
11	1.997 155	2.910 243	−0.201 108	0.803 798	1.783 769	1.792 883
12	1.999 321	2.936 268	−0.200 930	0.803 270	1.788 096	1.796 109
13	2.000 587	2.953 949	−0.200 792	0.802 839	1.791 041	1.798 111
14	2.001 321	2.966 228	−0.200 682	0.802 484	1.793 104	1.799 369
15	2.001 733	2.974 923	−0.200 594	0.802 190	1.794 586	1.800 165
16	2.001 948	2.981 188	−0.200 522	0.801 943	1.795 675	1.800 667
17	2.002 041	2.985 770	−0.200 462	0.801 735	1.796 491	1.800 979
18	2.002 058	2.989 170	−0.200 412	0.801 557	1.797 114	1.801 166
∞	2	3	−0.2	0.8	1.8	1.8
δ	$<5\times10^{-11}$	$<8\times10^{-10}$	$<2\times10^{-11}$	$<3\times10^{-11}$	$<3\times10^{-10}$	$<3\times10^{-10}$

[a] Also shown are corresponding exact values for $N\to\infty$ and the differences δ between the exact and extrapolated values (see text for details).

values and the first two terms in an $1/N$ expansion. The constant term $A_0 \approx 0.365315$ is the infinite-size limit of $-E_0(N)/N$, which was computed from the exact solution[28] by numerical integration. The form of the second term follows from conformal invariance[29,30] and involves the central charge $c = -22/5$ of our projected model. Apparently, there is no visible difference in the behavior of the ground-state energies for even or odd lengths of the chain.

Now let us turn to the excitations. We present numerical data of the lowest energy levels which in the finite-size scaling limit contribute to the partition functions $D_0^1(z, \bar{z}) = D_{3/2}^0(z, \bar{z}) = \chi_{1,1}(z)\,\chi_{1,1}(\bar{z})$, $D_0^2(z, \bar{z}) = D_{1/2}^0(z, \bar{z}) = \chi_{1,2}(z)\,\chi_{1,2}(\bar{z})$, and $D_1^2(z, \bar{z}) = D_{1/2}^4(z, \bar{z}) = \chi_{1,1}(z)\,\chi_{1,2}(\bar{z})$ [see Eq. (2.43)]. In what follows we will consider energy eigenvalues only, since the scaled momenta [see Eqs. (2.15) and (2.16)] of the levels (taking into account possible shifts of $N/2$ corresponding to a shift of π in the momentum) are always equal to their infinite-size limit (which clearly coincides for even and odd numbers of sites in the corresponding sectors). This

Table III. The Lowest Scaled Energy Gaps That Contribute to the Sectors D_0^2, resp. $D_{1/2}^0$, for Chains of Length N[a]

N	$\chi_{1,2}(z)\,\chi_{1,2}(\bar{z})$					
2	−0.449 035	0.200 814				
3	−0.416 278	0.487 387				
4	−0.408 337	0.555 038	1.051 478	1.766 633		
5	−0.405 135	0.577 995	1.314 348	1.719 483		2.552 737
6	−0.403 498	0.587 571	1.440 657	1.686 862	1.795 216	2.628 080
7	−0.402 541	0.592 214	1.505 180	1.665 366	2.084 961	2.649 283
8	−0.401 932	0.594 735	1.540 342	1.650 769	2.260 334	2.652 827
9	−0.401 519	0.596 227	1.560 652	1.640 489	2.369 054	2.650 289
10	−0.410 226	0.597 172	1.572 990	1.633 008	2.438 439	2.645 856
11	−0.401 011	0.597 804	1.580 817	1.627 406	2.484 026	2.641 109
12	−0.400 848	0.598 246	1.585 973	1.623 109	2.514 805	2.636 636
13	−0.400 721	0.598 567	1.589 479	1.619 743	2.536 112	2.632 626
14	−0.400 621	0.598 806	1.591 933	1.617 058	2.551 201	2.629 108
15	−0.400 541	0.598 990	1.593 693	1.614 885	2.562 107	2.626 049
16	−0.400 475	0.599 134	1.594 984	1.613 100	2.570 138	2.623 397
17	−0.400 420	0.599 248	1.595 949	1.611 617	2.576 150	2.621 097
18	−0.400 375	0.599 341	1.596 683	1.610 372	2.580 721	2.619 097
∞	−0.4	0.6	1.6	1.6	2.6	2.6
δ	$<3\times10^{-12}$	$<2\times10^{-11}$	$<2\times10^{-10}$	$<3\times10^{-12}$	$<5\times10^{-10}$	$<2\times10^{-13}$

[a] Also shown are corresponding exact values for $N \to \infty$ and the differences δ between the exact and the extrapolated values (see text for details).

also explains why we need not consider the sectors $D_1^3(z, \bar{z}) = D_{1/2}^1(z, \bar{z}) = \chi_{1,2}(z)\, \chi_{1,1}(\bar{z})$ because the corresponding energy spectra for finite chains are identical to those of the sectors $D_1^2(z, \bar{z}) = D_{1/2}^4(z, \bar{z}) = \chi_{1,1}(z)\, \chi_{1,2}(\bar{z})$ (the corresponding Hamiltonians are related by complex conjugation). Tables II and III show the lowest scaled energy gaps that contribute to the partition functions (2.43). Underlined values correspond to exactly degenerate eigenvalues. We also extrapolated the finite-size values to infinite length using the algorithm of ref. 31 (see also ref. 32), where the free variable of the extrapolation algorithm was chosen to be 2. The values of δ given in Tables II and III are the absolute difference between the extrapolated and exact scaling dimensions. Obviously, the agreement between extrapolated and exact data is extremely good. This observation is in perfect agreement with our conjecture that even in finite systems there is no destinction necessary between an even and an odd number of sites for this model. Hence, the half-integer charge sectors of the XXZ Heisenberg chain do not correspond to new boundary conditions here.

5. CONCLUSIONS

In ref. 1 it was shown that the finite-size scaling spectra of the XXZ Heisenberg chain with toroidal boundary conditions and an even number of sites contain the spectra of various series of models with central charge less than one, all of them belonging to the unitary series. A projection mechanism was presented that allowed for the explicit extraction of the spectra for each model in the finite-size scaling limit. The idea of the projection mechanism is first to choose an excited state of the (normalized) XXZ Heisenberg chain (2.11) with an anisotropy given by q and some boundary angle α_0 as the new ground-state of the projected system. The central charge of the projected model is related to q and α_0 by Eq. (2.23). In the next step we fix a specific relation between q and α_0, (2.27). This defines classes of models (R- and L-models) and their physical properties. Finally, we presented a subtraction algorithm, the projection mechanism which allows for the extraction of the finite-size scaling spectra of the projected systems from the XXZ Heisenberg chain by properly choosing charge sectors and boundary conditions and taking differences of sets of eigenvalues in these sectors. The levels that remain after this projection are the energy levels of the projected models.

It was observed that under certain circumstances, i.e., for certain choices of the anisotropy parameter q and the boundary twist α, the same mechanism also works on finite chains. This means that all the eigenvalues of certain sectors are exactly degenerate with part of the energy levels of the sectors from which the former are subtracted according to the projection rules derived in the finite-size scaling limit. Subsequently this important observation could be traced back to properties of the quantum algebra $U_q[sl(2)]$.[3]

In this paper we generalized these results in several aspects.

In the 1_R (1_L)-series of ref. 1, the low-temperature $O(p)$ models [$O(p)$ models] with central charge $c = 1 - 6/[m(m+1)]$ only the cases m even (odd) were considered. Here we completed this work by considering all values of m for both series, here called the $R = 2$ ($L = 2$) models. The missing models were shown to have the same sectors as the corresponding $R = 1$ (Potts) models [$L = 1$ (tricritical Potts) models] with the same central charge. The exact degeneracies observed in these models accounting for the possibility to apply the projection mechanism in some (not all) sectors even on finite chains were explained by the action of $U_q[sl(2)]$ (Section 3). This is an interesting generalization of the results of ref. 3, which only discusses what we call the $R = 1$ series. That the projection mechanism works exactly in finite chains in some sectors but not in all of them needs further understanding.

Furthermore, we generalized the projection mechanism to arbitrary real values of m, i.e., to all nonunitary minimal and nonminimal models with $c < 1$ in the R and L series. In this way we were able generate all possible sectors of these models according to their internal global symmetries and the resulting toroidal boundary conditions. As in the unitary series, the sectors of the nonunitary $R = 1, 2$ and $L = 1, 2$ models with the same central charge are in general different. This implies that the physical meaning of the operators in these systems is not specified by their anomalous dimension alone and hence universality classes are *not* completely defined by the central charge and some set of anomalous dimensions.

Extending the work of ref. 1, we considered chains with an odd number of sites giving rise to half-integer charge sectors. In these sectors we discovered in some of the projected models anomalous dimensions of so far unknown spinor fields. We studied the new sectors of the Ising model (2.46) and the 3-state Potts model (2.47) in more detail (Section 4.1). We showed that at their self-dual point these two models have an additional symmetry, the duality transformation. This symmetry gives rise to a new class of "duality twisted" toroidal boundary conditions. This means that the Hamiltonian of the corresponding models commutes with a generalized translation operator (4.10) performing a duality transformation at the boundary in combination with a translation. This is in close analogy to the well-known antiperiodic boundary conditions in the Ising model, where the generalized translation operator (4.4) performs a spin flip at the boundary combined with a translation. The form of the projection mechanism suggests that most of the projected systems have this additional symmetry.

However, we also discovered (nonunitary) systems where the spectra for integer and half-integer charge sectors are identical (Section 4.2). As the integer charge sectors include the sectors with periodic boundary conditions (in the projected system), these appear to be models where the duality twist is identical to periodic boundary conditions and, as a consequence, the duality transformation to the identity operator. This phenomenon is unknown in the unitary series.

Using the representation theory of the quantum algebra $U_q[sl(2)]$ and intertwining relations between the quantum algebra generators and different sectors of the Hamiltonian of the XXZ Heisenberg chain and the corresponding translation operator (see Appendix A), we were in fact able to explain *all* degeneracies observed numerically in ref. 1 and in addition we found similar symmetry properties for many of the new sectors (Section 3). In particular, for all sectors (including the half-integer charge sectors) of all $R=1$ models the projection procedure as described in this paper works for finite chains and the spectrum of the finite-size projected systems therefore is explicitly known. Furthermore, also the equality of the corresponding momentum eigenvalues of the degenerate levels could be proved (up to possible shifts of π in the momentum).

APPENDIX A. COINCIDENCES IN THE SPECTRA AND THE QUANTUM ALGEBRA $U_q[sl(2)]$

In what follows we show how the observed degeneracies in the spectra of the XXZ chain with various toroidal boundary conditions follow from the representation theory of the quantum algebra $U_q[sl(2)]$. This is achieved by establishing explicit intertwining relations between elements of $U_q[sl(2)]$ and different sectors of the XXZ Hamiltonian with toroidal boundary conditions. In addition, the equality of the momenta of the degenerate levels is proved by an analogous argument.

Let us commence by defining the quantum algebra $U_q[sl(2)]$. It is generated by the four generators $S^{\pm}$ and $q^{\pm S^z}$ subject to the relations (see ref. 3 and references therein)

$$q^{S^z}S^{\pm}=q^{\pm 1}S^{\pm}q^{S^z}, \qquad [S^+, S^-]=[2S^z]_q \tag{A.1}$$

(together with the relation $q^{S^z}q^{-S^z}=q^{-S^z}q^{S^z}=1$, which has been anticipated by the notation), where $[x]_q$, the "q-deformed of x," is defined by $[x]_q=(q^x-q^{-x})/(q-q^{-1})$ and q is a complex number with $q^2\neq 1$. A representation of this quantum algebra on $\mathscr{H}(N)\cong(\mathbb{C}^2)^{\otimes N}$ in terms of Pauli matrices is given by[3]

$$\begin{aligned} S^{\pm} &= \sum_{j=1}^{N} S_j^{\pm} = \frac{1}{2}\sum_{j=1}^{N} q^{(1/2)\sum_{k=1}^{j-1}\sigma_k^z}\sigma_j^{\pm} q^{-(1/2)\sum_{k=j+1}^{N}\sigma_k^z} \\ q^{S^z} &= q^{(1/2)\sum_{j=1}^{N}\sigma_j^z} \end{aligned} \tag{A.2}$$

The decomposition of this in general reducible representation on $\mathscr{H}(N)$ into irreducible representations of $U_q[sl(2)]$ has been discussed in ref. 3. It turns out that for generic values of q, i.e., q not being a root of unity, this decomposition resembles the undeformed $U[sl(2)]$ case,[33,34] whereas for q being a root of unity the situation is completely different.[35] Consider the case $|q| \neq 1$, where one has a continuous dependence on the complex variable q. If q approaches a root of unity, two formerly irreducible representations may become connected by the action of $U_q[sl(2)]$ constituting one larger indecomposable representation in this way. These "mixed" representations, however, are no longer irreducible, since they contain the smaller of the two representations as an irreducible submodule, i.e., the representation (A.2) in this case is no longer completely reducible.

Following an idea suggested in ref. 3, we now calculate the action of powers of the operators $S^{\pm}$ on the Hamiltonian $H(q, \alpha, N)$ of (2.1) and the translation operator $T(\alpha, N)$ of (2.4). The other generator q^{S^z} obviously commutes with both $H(q, \alpha, N)$ and $T(\alpha, N)$. By induction one can show that

$$\begin{aligned}
&\frac{(S^{\pm})^n}{[n]_q!} H(q, \alpha, N) \\
&\quad = H(q, q^{-2n}\alpha, N) \frac{(S^{\pm})^n}{[n]_q!} \\
&\quad \mp 2 \left(\sigma_N^z q^{-\sigma_N^z} S_1^{\pm} \frac{(q^{\pm 1} S^{\pm})^{n-1}}{[n-1]_q!} - S_1^{\pm} S_N^{\pm} \frac{(S^{\pm})^{n-2}}{[n-2]_q!} \right) (1 - q^{2(S^z \pm n)} \alpha^{\mp 1}) \\
&\qquad \mp 2 \left(\sigma_1^z q^{\sigma_1^z} S_N^{\pm} \frac{(q^{\mp 1} S^{\pm})^{n-1}}{[n-1]_q!} - S_N^{\pm} S_1^{\pm} \frac{(S^{\pm})^{n-2}}{[n-2]_q!} \right) (1 - q^{-2(S^z \pm n)} \alpha^{\pm 1})
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
&\frac{(S^{\pm})^n}{[n]_q!} T(\alpha, N) \\
&\quad = (q^{\sigma_1^z})^n \, T(\alpha, N) \frac{(S^{\pm})^n}{[n]_q!} \\
&\qquad + S_1^{\pm} \frac{(q^{\pm 1} S^{\pm})^{n-1}}{[n-1]_q!} T(\alpha, N)(1 - q^{2(S^z \pm n)} \alpha^{\mp 1}) \\
&\quad = \varepsilon T(q^{-2n}\alpha, N) \frac{(S^{\pm})^n}{[n]_q!} \\
&\qquad + S_1^{\pm} \frac{(q^{\pm 1} S^{\pm})^{n-1}}{[n-1]_q!} T(\alpha, N)(1 - q^{2(S^z \pm n)} \alpha^{\mp 1})
\end{aligned} \tag{A.4}$$

with $\varepsilon = \pm 1$. Here we used $[n]_q!$ for the product $[n]_q! = \prod_{j=1}^{n} [j]_q$. For the case $n=1$, Eq. (A.3) has already been obtained in ref. 3. The ambiguity in Eq. (A.4) is due to the square root [cf. the definition of the translation operator in Eq. (2.4)], the sign being fixed (but depending on the actual values of α, q, and n) once one chooses a particular branch. We do not want to investigate this further; we rather consider all momenta to be defined modulo π [which after all is exactly what enters in the finite-size scaling partition function; see Eqs. (2.13)–(2.15)].

It is our aim to extract from the above equations information about degeneracies in the spectra of $H(q, \alpha, N)$ for different boundary conditions α. For this purpose it is convenient to consider the several charge sectors separately. With the abbreviations

$$H_Q^K(N) = H(q, q^{2K}, N) \cdot \mathscr{P}_Q(N), \qquad T_Q^K(N) = T(q^{2K}, N) \cdot \mathscr{P}_Q(N) \tag{A.5}$$

we obtain from Eqs. (A.3) and (A.4) the following "intertwining relations":

$$\frac{(S^\pm)^n}{[n]_q!} H_{\pm Q}^{Q+n}(N) = H_{\pm(Q+n)}^{Q}(N) \frac{(S^\pm)^n}{[n]_q!} \tag{A.6}$$

$$\frac{(S^\pm)^n}{[n]_q!} T_{\pm Q}^{Q+n}(N) = \varepsilon T_{\pm(Q+n)}^{Q}(N) \frac{(S^\pm)^n}{[n]_q!} \tag{A.7}$$

for all $n = 1, 2, 3,...$ and $-N/2 \leqslant Q \leqslant N/2$, where $\varepsilon = \pm 1$.

Suppose now that the set

$$\left\{ v_j^Q \in \mathscr{H}_Q(N), j = 1, 2,..., \binom{N}{Q + N/2} \right\}$$

of common eigenvectors of $H_Q^K(N)$ and $T_Q^K(N)$ constitutes a basis of $\mathscr{H}_Q(N)$ and denote the set of pairs of corresponding eigenvalues by $\hat{\mathscr{E}}_Q^K(N)$,

$$\hat{\mathscr{E}}_Q^K(N) = \{(e_j, p_j) \mid H_Q^K(N)\, v_j^Q = e_j v_j^Q,\ T_Q^K(N)\, v_j^Q = \pm\exp(-ip_j)\, v_j^Q\} \tag{A.8}$$

Then, by applying Eqs. (A.6) and (A.7) to eigenstates of the Hamiltonian and the translation operator, one obtains the inclusions

$$\hat{\mathscr{E}}_{\pm Q}^{Q+n}(N) \supset \hat{\mathscr{E}}_{\pm(Q+n)}^{Q}(N) \tag{A.9}$$

for all positive values of Q and n. Strictly speaking, these inclusions are obtained for generic values of q only, but since the eigenvalues depend continuously on q (in the case of finite N), they also hold for the case that q is a root of unity. Indeed, in this case the observed degeneracies are much higher than for generic q.

To obtain the inclusions (A.9) we in fact also used the simple structure of the irreducible representations of $U_q[sl(2)]$ for generic q. One only has to realize that if one has a level in charge sector Q inside any irreducible representation, this representation contains exactly one level in the charge sectors Q' with $|Q'| \leqslant Q$, where Q' is integer (half-integer) if Q is integer (half-integer).

Using in addition the charge conjugation transformation C of (2.9) which leads to $\hat{\mathscr{E}}^K_Q(N) \equiv \hat{\mathscr{E}}^{-K}_{-Q}(N)$ for all values of K and Q, one finally obtains the following result:

$$\hat{\mathscr{E}}^K_Q(N) \supset \hat{\mathscr{E}}^Q_K(N) \qquad \text{for all} \quad |K| \geqslant |Q| \tag{A.10}$$

and K and Q are both integer (half-integer) numbers for even (odd) number of sites, respectively.

Now we turn to the case when q is a root of unity. We define an integer p by

$$p = \min\{n > 0 \mid q^{2n} = 1\} = \min\{n > 0 \mid q^n \in \{\pm 1\}\} \tag{A.11}$$

In this case the boundary condition is defined modulo p, since $H^K_Q(N) = H^{K+\nu p}_Q(N)$ and $T^K_Q(N) = T^{K+\nu p}_Q(N)$ for all $\nu \in \mathbb{Z}$. Hence from Eq. (A.10) one directly obtains the result

$$\hat{\mathscr{E}}^K_Q(N) \supset \hat{\mathscr{E}}^Q_{K+\nu p}(N) \tag{A.12}$$

for all $K \geqslant |Q|$, where ν is an arbitrary positive integer number and we restrict K to the values $0 \leqslant K \leqslant p - 1/2$.

In addition to this rather trivial modification one observes additional degeneracies which show up due to the mixing of irreducible representations of $U_q[sl(2)]$ when q approaches a root of unity, since all states in an indecomposable representation that correspond to eigenstates of our Hamiltonian in the appropriate sectors have the same energy (and momentum). To make the statement more precise, we use the results of ref. 3 on the indecomposable representations of $U_q[sl(2)]$ for q being a root of unity. This finally leads to the inclusion relation

$$\hat{\mathscr{G}}^K_Q(N) \supset \hat{\mathscr{G}}^Q_K(N) \equiv \hat{\mathscr{G}}^{-Q}_{-K}(N) \tag{A.13}$$

for $p/2 \geqslant |K| \geqslant |Q| \geqslant 0$ and $2K \equiv 2Q \equiv N \bmod 2$, where $\hat{\mathscr{G}}^K_Q(N)$ is defined to be

$$\hat{\mathscr{G}}^K_Q(N) = \bigcup_{\nu \in \mathbb{Z}} \hat{\mathscr{E}}^K_{Q+n\nu}(N) \tag{A.14}$$

By closer inspection of Eq. (A.13) one realizes (see ref. 3) that the set of all levels contained in $\hat{\mathscr{G}}_Q^K(N)$ which are not contained in $\hat{\mathscr{G}}_K^Q(N)$, i.e., the difference $\hat{\mathscr{D}}_Q^K(N)$ of the two sets

$$\hat{\mathscr{D}}_Q^K(N) = \hat{\mathscr{G}}_Q^K(N) - \hat{\mathscr{G}}_K^Q(N) \tag{A.15}$$

is just the set of levels that correspond to the highest weights of those irreducible representations in the decomposition of the representation (A.2) which are isomorphic to $U[sl(2)]$ irreducible representations. It is therefore obvious that Eq. (A.14) really relies on the structure of the irreducible representations and contains more information about the spectrum than Eq. (A.12).

APPENDIX B. THE DUALITY TRANSFORMATION IN THE ISING QUANTUM CHAIN

Here we give a brief review of the duality transformation in the Ising quantum chain. The duality transformation is a map between the low- and high-temperature phases of the system. Denoting the inverse temperature by λ and the Hamiltonian (to be specified below) by $H(\lambda)$, one finds that $H(\lambda)$ and its dual $H^D(\lambda)$ have the same spectrum[22] and are related by the duality transformation D

$$DH(\lambda) = H^D(\lambda)\, D \tag{B.1}$$

D depends on the boundary conditions. In what follows we consider the mixed sector Hamiltonian $\bar{H}$ of a chain with M sites,

$$\bar{H}^{(\pm)}(\lambda) = -\left\{ \sum_{j=1}^{M-1} (e_{2j-1} - \tfrac{1}{2}) + \lambda(e_{2j} - \tfrac{1}{2}) \right.$$
$$\left. + (e_{2M-1} - \tfrac{1}{2}) + \lambda(e_{2M}^{(\pm)} - \tfrac{1}{2}) \right\} \tag{B.2}$$

where the e_j for $1 \leqslant j \leqslant 2M-1$ are defined by Eq. (4.6) and $e_{2M}^{(\pm)} = (1 \pm S\sigma_M^z \sigma_1^z)/2$. Here S is the spin-flip operator defined in Eq. (4.2). In $\bar{H}^{(+)}$ one has periodic boundary conditions in the even charge sector ($S = 1$) (for brevity, we denote the projection on this sector by H_0^0) and antiperiodic boundary conditions in the odd sector ($S = -1$) denoted by H_1^1. $\bar{H}^{(-)}$ corresponds to periodic boundary conditions in the odd sector (denoted H_1^0) and antiperiodic boundary conditions in the even sector H_0^1. $\bar{H}^{(\pm)}(\lambda)$ satisfies the duality relation

$$D^{(\pm)} \bar{H}^{(\pm)}(\lambda) (D^{(\pm)})^{-1} = \bar{H}^{(\pm)^D}(\lambda) = \lambda \bar{H}^{\pm}\left(\frac{1}{\lambda}\right) \tag{B.3}$$

The duality transformation $D^{(\pm)}$ is given by[24,25]

$$D^{(+)} = \prod_{j=1}^{2M-1} g_j, \qquad D^{(-)} = D^{(+)}\sigma_M^z \tag{B.4}$$

The operators g_j are related to the e_j by $g_j = (1+i)\,e_j - 1$. The g_j are invertible (one finds $g_j^{-1} = g_j^*$) and one can show that D acts on the e_j as follows[25,26]:

$$\begin{aligned} D^{(\pm)}e_j(D^{(\pm)})^{-1} &= e_{j+1}, \qquad 1 \leqslant j \leqslant 2M-2 \\ D^{(\pm)}e_{2M-1}(D^{(\pm)})^{-1} &= e_{2M}^{(\pm)} \\ D^{(\pm)}e_{2M}^{(\pm)}(D^{(\pm)})^{-1} &= e_1 \end{aligned} \tag{B.5}$$

The e_{2j} are the operators dual to the e_{2j-1}.

Using the projectors $P_{(\pm)} = (1 \pm S)/2$ on the even and odd sectors of $\bar{H}^{(\pm)}$, $P_{(\pm)}\bar{H}^{(\pm)} = H_l^{l'}$ as defined above, one finds the duality relations for the sectors of the Ising Hamiltonian

$$H_l^{l'D}(\lambda) = \lambda H_l^{l'}\left(\frac{1}{\lambda}\right) = H_{l'}^{l}(\lambda) \tag{B.6}$$

with l, $l' = 0, 1$. In the derivation of these relations one has to take into account that $D^{(+)}$ commutes with $P_{(\pm)}$, but $D^{(-)}P_{(\pm)} = P_{(\mp)}D^{(-)}$. We see that the duality relation (B.3) does not hold for the Hamiltonian H of (4.1) given in Section 4, which in terms of the sectors $H_l^{l'}$ is given by $H = H_0^0 + H_1^0$ for periodic boundary conditions and $H = H_0^1 + H_1^1$ for antiperiodic boundary conditions.

From relations (B.5) it is obvious that D^2 commutes with $\bar{H}^{(\pm)}$ and is related to the translation operators T of (4.3) and T' of (4.4). A short calculation shows

$$\begin{aligned} T^{(+)} &= (D^{(+)})^2 = i^{M+1}T(P_{(+)} + \sigma_M^x P_{(-)}) \\ T^{(-)} &= (D^{(-)})^2 = i^{M}T(P_{(-)} + \sigma_M^x P_{(+)}) \end{aligned} \tag{B.7}$$

Since $(D^{(-)})^2$ commutes with $P_{(\pm)}$, we can construct a translation operator commuting with the H of (4.1) by taking suitably chosen projections on the even and odd subspaces. We find

$$T = i^{-M-1}(T^{(+)}P_{(+)} + iT^{(-)}P_{(-)}) \tag{B.8}$$

$$T' = T\sigma_M^x = i^{-M-1}(T^{(+)}P_{(-)} + iT^{(-)}P_{(+)}) \tag{B.9}$$

commuting with H with periodic boundary conditions and antiperiodic boundary conditions, respectively. The factors i^M, i^{M+1} are of course irrelevant.

Next we consider the mixed sector version of the Ising Hamiltonian $\tilde{\tilde{H}}(\lambda, \mu)$ defined by

$$\tilde{\tilde{H}}^{(\pm)}(\lambda, \mu) = -\left\{ \sum_{j=1}^{M-1} (e_{2j-1} - \tfrac{1}{2}) + \lambda(e_{2j} - \tfrac{1}{2}) + \mu(e_{2M-1}^{(\pm)} - \tfrac{1}{2}) \right\} \quad \text{(B.10)}$$

where the e_j for $1 \leqslant j \leqslant 2M-2$ are defined as in Eq. (4.6), but $e_{2M-1}^{(\pm)} = (1 \mp S\sigma_M^y \sigma_1^z)/2$. The duality transformation $\tilde{D}^{(\pm)}$ satisfying analogous relations to (B.5) (with $2M-1$ replaced by $2M-2$ and $2M$ replaced by $2M-1$) is given by[25,26]

$$\tilde{D}^{(+)} = \prod_{j=1}^{2M-2} g_j, \qquad \tilde{D}^{(-)} = \tilde{D}^{(+)} \sigma_M^x \quad \text{(B.11)}$$

From this one obtains the transformed Hamiltonian satisfying

$$\tilde{D}^{(\pm)} \tilde{\tilde{H}}^{(\pm)}(\lambda, \lambda)(\tilde{D}^{(\pm)})^{-1} = \lambda \tilde{\tilde{H}}^{(\pm)}\left(\frac{1}{\lambda}, 1\right) \quad \text{(B.12)}$$

Since $\tilde{D}^{(\pm)}$ commutes with S, (B.12) holds for each sector separately. It is important to note that $(\tilde{D}^{(\pm)})^2$ *does not commute with* $\tilde{\tilde{H}}^{(\pm)}(\lambda, \mu)$ *unless* $\lambda = \mu = 1$. Only in this case $\tilde{\tilde{H}}^{(\pm)}$ is translationally invariant with $\tilde{T}^{(\pm)} = (\tilde{D}^{(\pm)})^2$ given by

$$\begin{aligned} \tilde{T}^{(+)} &= i^M T(P_{(+)} g_{2M-2}^* g_{2M-1}^* - iP_{(-)} g_{2M-2} g_{2M-1}) \\ \tilde{T}^{(-)} &= i^M T(P_{(+)} g_{2M-2} g_{2M-1} + iP_{(-)} g_{2M-2}^* g_{2M-1}^*) \end{aligned} \quad \text{(B.13)}$$

where g_{2M-2} and g_{2M-1} are defined in Eq. (4.11). By taking projections on the even and odd subspaces one obtains the translation operator $\tilde{T}$ of (4.10) commuting with $\tilde{H}$ given in (4.5):

$$\tilde{T} = T g_{2M-2} g_{2M-1} = (-i)^M (T^{(-)} P_{(+)} + i T^{(+)} P_{(-)}) \quad \text{(B.14)}$$

Finally, we compute $\tilde{T}^M$. We define $g_0 = -(1-i)(1 - i\sigma_M^z \sigma_1^z)/2$. It follows from Eq. (B.5) that $(D^{(+)})^{-1} g_0 D^{(+)} = -(1-i)(1 - iS\sigma_M^x)/2$ and therefore

$$\begin{aligned} \tilde{T}^M &= g_0 \prod_{j=1}^{2M-1} g_j = -\frac{1-i}{2} \tilde{D}^{(+)}(1 - iS\sigma_M^x)\, g_{2M-1} \\ &= -\tilde{D}^{(-)} P_{(+)} - i\tilde{D}^{(+)} P_{(-)} \end{aligned} \quad \text{(B.15)}$$

This is the duality transformation in the even and odd sectors of $\tilde{H}$.

We want to conclude this short review by noting that the duality transformation in the 3-state Potts model proceeds along similar lines. Only the operators e_j have to be replaced by those given in Section 4 for the 3-state Potts model and the discussion of the various sectors is slightly more complicated due to the higher symmetry (S_3 instead of S_2). In particular, the mixed sector version of the Hamiltonian (4.12) is defined by taking $\sigma_{M+1} = \omega^{\kappa} Z \sigma_1$, $\kappa = 0, 1, 2$.

ACKNOWLEDGMENTS

U.G. would like to thank M. Baake, B. Davies, O. Foda, B. M. McCoy, P. A. Pearce, V. Rittenberg, M. Scheunert, and Y.-K. Zhou for valuable discussions. We gratefully acknowledge Research Fellowships of the Deutsche Forschungsgemeinschaft (DFG) (U.G.) and the Minerva foundation (G.S.).

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